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ELEMENTARY MATHEMATICS

COMPRISING ALGEBRA, TRIGONOMETRY AND
CO-ORDINATE GEOMETRY

FOR

INTERMEDIATE PHYSICS STUDENTS

by

SHIV CHARAN SINGH, M.Sc., P.E.S.
Ex-Principal Government College, Dharamsala

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PREFACE

This book has been written to meet the requirements of the Physics students of the Punjab University appearing for the Intermediate Examination. There existed no single book giving the elementary portions of Algebra, Trigonometry and Co-ordinate Geometry, therefore great difficulty was being felt by teachers as well as students in taking out of so many different books exactly what was needed. It is hoped that this book will remove this difficulty.

An attempt has been made to stick to the syllabus, but a few things have been added here and there, because their knowledge has been considered useful for a student of Physics. Emphasis has been laid on the use of Logarithms by employing them wherever convenient in the solved examples.

The chapter on approximations is meant to acquaint the student with the nature of approximations so often used in Physics. A friend of mine told me that as a student of Physics he was always puzzled to see why one approximation was used and not the other. Logarithms have been given a place almost in the beginning of the book, because it has been considered necessary that the student should begin practising them as soon as he starts his work. Graphs have been put in a separate chapter on account of their importance. Some general observations on drawing graphs have been given to indicate the right method of drawing them. It has been noticed that majority of students draw false or microscopic curves, because the rules found in books on practical physics being not properly understood at this stage are never read with as much attention as they deserve.

My best thanks are due to my respected teacher and colleague, Lala Hukam Chand, M.A., B. Sc., Professor of Mathematics, for his kindness in going through the manuscript.

Mr. D. P. Gupta M.Sc. has helped me in revising the manuscript and has corrected the proofs of this edition for which I am specially thankful to him.

Articles marked with asterisks^{+} may be omitted.

SHIV CHARAN SINGH.

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ELEMENTARY MATHEMATICS

CHAPTER I

GRAPHS

1. A graph is a pictorial representation of the variation of one quantity with respect to another. Just as no amount of description can depict a natural scenery as vividly and as truly as a picture, so nothing can show as clearly and as faithfully the changes in one quantity with the corresponding changes in the other, as a graph.

2. The student is probably already familiar with the method of representing by means of a graph the variation of one quantity related to another, or the variation of one quantity whose value depends upon another. We take two perpendicular straight lines X′OX and Y′OY intersecting in the point O. These are called respectively the *x-axis* and the *y-axis* and O is called the *origin*. X′OX and Y′OY taken together are called the *axes of co-ordinates*. Distances from O measured along OX and OY respectively are called the *abscissae* and the *ordinates*. Distances from O are to be taken as positive, when they are measured towards the right from O or in the upward direction from O, and as negative when they are measured in the opposite directions. We then represent the values of one quantity along the x-axis according to a suitable scale of measurement, and then represent the corresponding values of the second quantity along the y-axis. We then plot the points corresponding to each pair of values of the two related quantities. All the points so obtained are then joined by a smooth curve, free of corners. This gives us the graph representing the variation of one quantity related to another. The student must bear in mind the following important points when drawing a graph from the observations of an experiment :—

1. The number of observations should not be less than five, the more the batter.
2. The range of observations should be as wide as possible.
3. The quantity which is known beforehand, is predetermined or is simpler, should form the abscissa, and the other, the ordinate.
4. Scales chosen along the axex to represent the abscissae and the ordinates should be liberal enough to give a fairly large curve.

5. The origin, *viz.*, the point where the axes meet should be fixed at the left hand lower corner of the page and need not represent the zeros of the scales. The scales should rather begin at some round figures just less than the smallest values of the quantities in the set of observations, except when extrapolation is required, in which case the required value must be kept in the scale.

6. The scales must be marked in the increasing order towards the positive directions (right and upwards).

7. The observations should be written in a regular ascending order of the abscissa before the plotting is begun.

8. The plotted points should be joined by a regular free hand curve and not by zigzag lines. The curve should pass between all the marked points and not necessarily through them.

Example. Draw a graph between the readings of a Centigrade and a Fahrenheit thermometer as recorded below :—

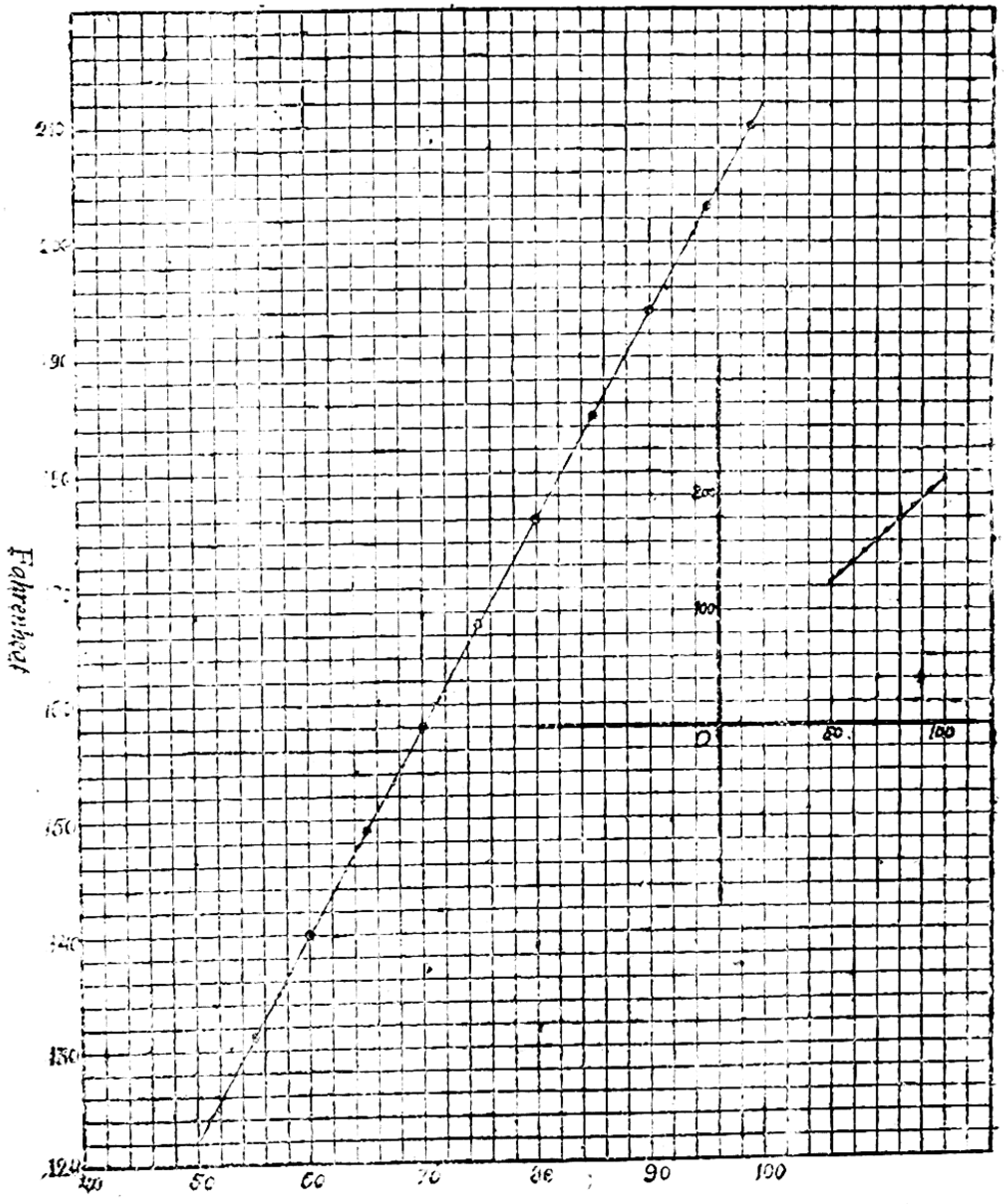
C°	99	95	90	85	80	75	70	65	60	55	50
F°	210	203	194	185	176	167	158	149	140	131	122

Two lines have been drawn at right angles to each other meeting at the left lower corner of the graph paper (see page 3). There are about 50 centigrade divisions to be represented along the axis of x and the number of divisions available on the paper is just more than half this number. The most convenient scale will thus be one in which one division represents 2°C. Similarly there are about 90°F in our set of observations and the number of divisions available along the axis of y is also a little more than half of it. A similar scale having one division equal to 2°F will be suitable along the axis of y. The origin has been denoted as 40°C for the axis of x and 120°F for the axis of y. The points (50, 122), (55, 131) etc. have been plotted and joined by a free hand curve. This curve represents the required graph.

The result of failing to observe any one of the rules given above, will be to get abnormally small, meaningless or false curves The graph in the inset (see the figure on page 3) shows how a ridiculously small curve is obtained when scales chosen are not liberal enough.

Interpolation and Extrapolation

3. When a graph is drawn from a given set of observations or data, the value of one quantity corresponding to any

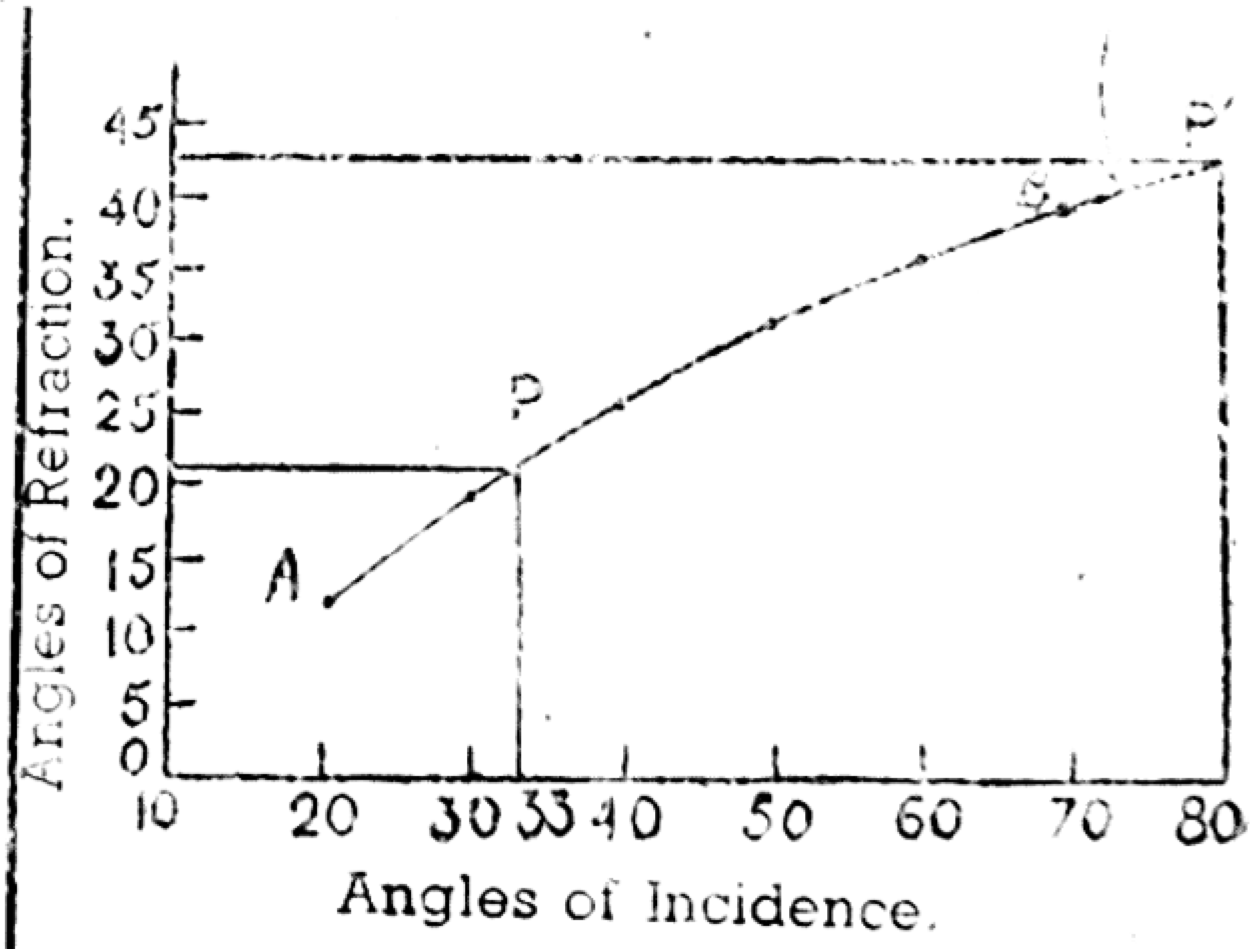


given value of the other can be read off from the graph. If the required value falls within the curve drawn from the given data, the value is said to have been found by **Interpolation,** but if the curve is to be extended one way or the other to get the required value, it is said to have been found by **Extrapolation.**

Centigrade

Example. Angles of incidence and refraction of rays of light passing from air into glass are observed by an experiment and tabulated below. Find the angles of refraction of rays whose angles of incidence are 33° and 80°.

i	r
20°	13°·2
30°	19°·5
40°	25°·4
50°	30°·7
60°	35°·3
70°	38°·8



Take angles of incidence as abcissae and angles of refraction as ordinates and draw the graph AB as in the above figure.

To find the angles of refraction corresponding to 33° and 80°, draw prependiculars to the axis of x at these points.

The perpendicular from 33° meets the graph at P. Read the ordinate of P. This gives the value of the coruesponding angle of refraction.

The perpendicular from 80° does not meet the graph, hence the graph has been produced (retaining so far as possible its curvature near its last portion) to meet it at P′. The ordinate of P′ gives the value of the angle of refraction corresponding to 80°.

The ordinate of P=21·5 and that of P′=43.

Point P lies on the graph and P′ has been found by producing it, therefore the value 21·3 corresponding to 33° has been found by Interpolation and the value 43 corresponding to 80° by Extrapolation.

4. By Interpolation the value found is generally correct, because the point lies within the range of our observations and unless the curve undergoes an abrupt change between the two observations surrounding the point, we are more or less certain about its path.

In Extrapolation there are greater chances of error, because it is quite possible that the path of the curve beyond our last observation may be quite different from the one found by producing the curve and thus the true position of P′ may be somewhere else.

The example given above illustrates this point quite well. Beyond 70° the actual graph bends more rapidly, to become almost parallel to the axis of x at 90°. but we produce it to retain its curvature between 60° and 90°, which portion is yet fairly steep.

Actual value of the angle of refraction corresponding to 33°= 21°·8 while that corresponding to 80°=41°·1.

The value found by interpolation is fairly correct while that found by extrapolation though not so is yet a fair index of the true value.

EXAMPLES 1

1. Draw a graph between natural numbers and their squares. Find the square of 2·3 from the graph.

2. Draw a graph between natural numbers and their reciprocals.

3. The following readings for the E. M. F. and current through a given circuit were observed :—

Volts	3·4	10·2	15·3	25·5	30·6
Amperes	0·2	0·6	0·9	1·5	2·0

Plot a graph, (P. U.)

4. With an India rubber rod hung at its upper end and loaded at the lower one, the following series of readings were taken :

Load	4 oz.	8 oz.	12 oz.	1 lb.	$1\frac{1}{2}$ lb.
Observed length	28·4″	28·61″	28·8″	28·98″	29·4″

Plot a curve showing how the length varies with the load. (P.U.)

5. The following observations were made with a simple pendulum :—

Length	84·3	93·8	101·5	112·6	125·8 cm.
Period	1·84	1·94	2·02	2·13	2·25 sec.

Draw a graph to show the relation between the length and the square of the period. Find the length of Second's pendulum from the curve. (A Second's pendulum is one which goes from one extremity to the other in one second).

6. The following quantities of heat given to 5 gms. of a solid at its melting point (0°C) are found to melt it, and raise the liquid produced to the temperatures given below :—

Therms	410	443	492	545	608	647
Temp.	3°	10°	19°	30°	43°	50°C.

Prepare a smooth graph to show how the heat required varies with the temperature, and calculate from the graph the latent heat and specific heat of the liquid. (P.U. 1929)

[**Hint.** The graph is a straight line meeting the x-axis at about 395 therms point showing that 5 gms. of the solid take this amount of heat for melting only. An ordinate drawn anywhere from the graph shows the rise of temp. in 5 gms. of the liquid, produced by the number of therms represented by the distance on the x-axis from 395 therms point to the foot of the ordinate.]

CHAPTER II

APPROXIMATIONS AND ERROR PERCENTAGES

1. APPROXIMATIONS

5. Sciences depend entirely upon their practical work. New facts are found as results of observations and experiments, and they are recorded in a systematic manner. Rapid progress in all the sciences in the last quarter of a century has been due solely to the starting of so many experimental institutes. Practical work in Physics, Astronomy, Chemistry chiefly and to some extent in all other sciences consists of measurements by instruments of various degrees of sensitiveness and calculations from those observations. Mathematics is a theoretical subject, but its chief importance lies in its applications to practical sciences. In fact, it affords a key to most of the very intricate practical problems.

In calculations one can greatly simplify the processes and save time by using approximations, without any appreciable loss in the accuracy. On the other hand, if one does not use approximations, one will be giving some absurd or at least useless figures in the result.

Illustration. Suppose we measure the side of a square by a metre rod and calculate its area.

Metre rods are divided into centimetres and millimetres, and the length measured, is, say, 4 cm., 3 mm., plus a fraction of a millimetre. This fraction is generally judged by the eye, therefore one cannot be quite certain of its accuracy. If one reads this fraction as $\frac{4}{10}$th. of a millimetre it is possible that it may actually be $\frac{3}{10}$th or $\frac{5}{10}$th. Thus the side of a square has been measured to be 4·34 cm. and the second place of decimals (4) is not trustworthy.

Area calculated from this observation $=(4{\cdot}34)^2=18{\cdot}8356$ square centimetres.

Let the true value of the side of the square be $=4{\cdot}33$ cm.

True value of the area $=(4{\cdot}33)^2=18{\cdot}7489$ square centimetres.

In the result obtained from our experiment even the first place of decimals (8) is not correct. It is greater than the true value by one. Figures in the second, third and fourth places of decimals are altogether untrustworthy. In order to determine which figures to keep and which to reject we will first find how much error we introduce by rejecting them.

II. ERROR PERCENTAGES

6. It is a common practice in our everyday dealings, that if there are two quantities, one small and the other comparatively very large we do not care much for the smaller one. For instance, if we buy some thousands of maunds of a commodity, we will not at all mind a difference of a few chhataks or even a few seers one way or the other, but if it is a question of a few seers only, the chhattaks are also very important. Similar is the case in scientific experiments. In the illustration of the last article our observation is quite correct in the first place of decimals, but the second place is only correct within a range of $\pm\frac{1}{10}$ mm. In calculating the amount of error he *relative importance* of the error is the main consideration. The error of one anna in thousands of rupees is trifling compared with the same error of one anna in a few annas. The absolute value of the error is thus meaningless. A small fraction of a pie may have some tangible value for a student of pure mathematics but it does not have any meaning for a practical businessman. In order to have a uniform basis, the errors are calculated in percentages. An error of 1 mm. in 2 cm. (20 mm.) is equivalent to an error of 1 in 20 or 5 in a hundred, *viz.*, 5%.

Example. Calculate the error percentage in the illustration of the last article.

Area found by experiment $=18{\cdot}8356$ sq. cm.

True area $=18{\cdot}7489$ sq. cm.

Net error $=0{\cdot}0867.$

There is an error of 0·0867 in 18·7489

$$\therefore \quad \text{Error percentage}=\frac{0{\cdot}0867\times 100}{18{\cdot}7489}=0{\cdot}47\%.$$

7. (*a*) If we use an approximation and keep our result up to three places of decimals only,

the calculated result $=18{\cdot}836.$

(The fourth decimal figure [6] has been rejected but as this figure is more than 5 the third figure has been increased by one. In case the figure rejected is 5 or more the previous figure is increased by one and if it is less than 5 no change is made.)

The true result $=18{\cdot}7489$

$$\therefore \quad \text{error}=\frac{0{\cdot}0871}{18{\cdot}7489}\times 100=0{\cdot}47\%$$

(*b*) On using a further approximation and keeping our result up to two decimal places only,

the calculated result $=18{\cdot}84$

The true result $=18{\cdot}7489$

$$\text{error}=\frac{0{\cdot}0911}{18{\cdot}7489}\times 100=0{\cdot}49\%.$$

The student might have noticed that in an approximation up to three places of decimals the error percentage remains the same while in the approximation up to two places of decimals the error percentage is only 0·02 per cent more.

If we can save a great amount of labour and time by making approximations and we do not add materially to the error percentage, we gain a great deal and lose nothing. Thus it will be very economical to use approximations.

To a scientist the figure in the unit's place in 83452 is as unimportant as the fifth decimal place in ·83452 because the error percentage in omitting them is the same in both cases although it means neglecting of 2 in the first case and of 0·00002 in the second.

8. We know that $(1+a)^2=1+2a+a^2.$

When a is very small, a^2 will be much smaller still, therefore negligible in comparison with the remaining terms.

Thus $(1+a)^2 = 1+2a$ approximately.

The following examples will illustrate the significance of this approximation.

Example. Calculate the error percentage involved in using the approximation $(1+a)^2=1+2a$ when $a=0{\cdot}012$.

Approximate value $=1+2\times 0{\cdot}012=1{\cdot}024$.

True value $=1+2\times 0{\cdot}012+(0{\cdot}012)^2=1{\cdot}024144$.

Net error $=0{\cdot}000144$.

Error percentage $=\dfrac{0{\cdot}000144}{1{\cdot}024144}\times 100=0{\cdot}01\%$.

This error percentage is hardly appreciable.

Rule. *Whenever a quantity is equal to the sum or difference of other quantities, some of which are extremely small compared with the others, the smaller ones may be neglected.*

Caution. We cannot neglect a small quantity if it multiplies or divides another.

For instance, if we omit $\dfrac{1}{10000}$ in $84567\times\dfrac{1}{10000}$ we get the result 84567 instead of 8·4567 which is evidently absurd, because it is ten thousand times the correct result.

9. It is not necessary here to describe the detailed arithmetical processes for approximations, because it is a mere waste of time to make long calculations in practical sciences by direct processes of arithmetic. A very convenient method of calculations is given in the next chapter on logarithms. This automatically contains approximations.

SOLVED EXAMPLES

Example 1. The true length of a straight line is 10·23 cm. and it is measured by an experimenter to be 10·22 cm. Calculate the error percentage in (a) the actual observation, (b) the area of a square, and (c) the volume of a cube having the given line as its side.

(a) The error in the observation $=0{\cdot}01$

Error percentage $=\dfrac{0{\cdot}01}{10{\cdot}23}\times 100=0{\cdot}098\%.$

(*b*) Area calculated from this observation

$=(10{\cdot}22)^2=104{\cdot}4484$ sq. cm.

True area $=(10{\cdot}23)^2=104{\cdot}6529$ sq. cm.

Error percentage $=\dfrac{0{\cdot}2045}{104{\cdot}6529}\times 100=0{\cdot}196\%$

(*c*) Calculated volume $=1067{\cdot}463$ c.c.

True volume $=1070{\cdot}599$ c.c.

Error percentage $=\dfrac{3{\cdot}136}{1070{\cdot}599}\times 100=0{\cdot}293\%.$

It will be noticed that in the square, the error percentage is just about double, and in the cube it is about three times that in the original observation.

Example 2. Show that $\dfrac{1}{1-x}=1+x$ approximately when x is very small.

By actual division $\dfrac{1}{1-x}=1+x+x^2+x^3\ldots\ldots..;$ x is small, therefore x^2, x^3, etc. are negligible.

Hence $\dfrac{1}{1-x}=1+x$ approximately.

EXAMPLES II

1. $(1+\beta)=(1+\alpha)^2$. Show that $\beta=2\alpha$ approximately when α is very small. Calculate the error percentage involved in this approximation when $a=0{\cdot}000012$. (Co-efficients of superficial and linear expansion).

2. $1+\gamma=(1+\alpha)^3$. Show that $\gamma=3\alpha$ approximately. Find the error percentage in this approximation when α is the same as in the last question (co-efficients of cubical and linear expansion).

3. $1+\alpha t=(1+\beta t)(1+gt)$. Show that $\alpha=\beta+g$, when β and g are very small and t is not very large. (Co-efficients of real and apparent expansions.)

4. Show that $\frac{1}{1+x}=1-x$ approximately, when x is very small.

5. How much error percentage is involved in putting $\sqrt{1+x}=1+\frac{1}{2}x$ when the value of $x=0{\cdot}006009$?

6. $\dfrac{(m+w)\frac{T}{t}-w}{T}=\dfrac{m}{M}\times\dfrac{T}{t}$ approximately when w is small. Calculate the error percentage when $m=120$ gm., $M=96$ gm., $T=5$ minutes, $t=6$ minutes and $w=4$ gm. (Specific heat by cooling method).

CHAPTER III

LOGARITHMS

10. Logarithms provide a very useful tool in the hands of a scientist for making calculations, correct up to the required degree of accuracy, in the shortest possible time. They are as useful and indispensable for a student of science as arithmetic is for a shopkeeper.

11. Suppose $a^x=n$ then x is called the logarithm of n to the base a. It is written as $\log_a n$

$\log_a n=x$ expresses the same relation as $a^x=n$.

Definition. The **Logarithm** of a number to a given **base** is the **index** of the power to which the *base* must be raised to be equal to the given number.

$\log_3 81=4$ because $3^4=81$.
$\log_2 32=5$ $\qquad 2^5=32$.

GENERAL RELATIONS

12. (1) **The logarithm of unity to any base is zero,** because any quantity raised to the power zero is equal to 1.

$$a^0=1, \quad \therefore \log_a 1=0.$$

(2) **The logarithm of the base itself is 1.**

$$a^1=a, \quad \therefore \log_a a=1.$$

(3) **The logarithm of the product of two or more numbers is equal to the sum of their logarithms.**

Let $\log_a m=x$; $\log_a n=y$; $\log_a p=z$;............

then $m=a^x$; $n=a^y$; $p=a^z$;............(By definition)

$$m\times n\times p\times\ldots\ldots\ldots=a^x\times a^y\times a^z\times\ldots\ldots=a^{x+y+z+\ldots}$$

$$\therefore\ \mathbf{log}_a\,(\mathbf{m}\times\mathbf{n}\times\mathbf{p}\times\ldots\ldots)=x+y+z\times\ldots\ldots\text{(By definition)}$$

$$=\mathbf{log}_a\,m+\mathbf{log}_a\,n+\mathbf{log}_a p+\ldots\ldots$$

(4) **The logarithm of a fraction is equal to the difference between the logarithms of the numerator and the denominater.**

Let $\log_a m=x$ and $\log_a n=y$.

then $m=a^x$ and $n=a^y$

$$\frac{m}{n}=\frac{a^x}{a^y}=a^{x-y}$$

$$\therefore\ \mathbf{log}_a\left(\frac{m}{n}\right)=x-y=\mathbf{log}_a\,m-\mathbf{log}_a\,n.$$

(5) **The logarithm of any power** (integral or fractional) **of a number is equal to the product of the index of the power and the logarithm of the number.**

Let $\log_a m=x$

then $m=a^x\ \therefore\ m^r=a^{rx}$

$$\log_a m^r=rx=r\log_a m$$

$$\therefore\ \mathbf{log}_a\,m^r=r\,\mathbf{log}_a\,m.$$

13. The logarithm of a number to any base is not usually an integer. It consists generally of an integral part and a fractional part expressed in decimals. Further, as will be seen below, the logarithm of any number less than one to a base greater than one is negative. In such a case it is convenient for purposes of tabulation that the fractional part of the logarithm be kept positive. Thus, if the logarithm of any number be $-1{\cdot}4367$, we write it as

$$-1{\cdot}4367=-1-0{\cdot}4367=-2+1-0{\cdot}4367=-2+0{\cdot}5633$$

$$=\bar{2}{\cdot}5633.$$

The bar above 2 denotes that 2 is to be taken with the negative sign and the decimal part ·5633 is to be taken with the positive sign. By this convention the logarithms of all numbers are expressed in such a way that the decimal part of the logarithm is always positive ; the integral part is positive or negative according as the logarithm is positive or negative.

Def. The fractional part of the logarithm after being made positive is called the *mantissa*, and the integral part of the logarithm, when the fractional part has been made positive, is called the *characteristic*.

Examples. (*i*) Logarithm$=2·3584$, mantissa$=·3584$, characteristic$=2$; (*ii*) logarithm$=0·4761$, mantissa$=·4761$, characteristic$=0$; (*iii*) logarithm$=-0·7685=\bar{1}·2315$, mantissa$=·2315$, characteristic$=-1$; (*iv*) logarithm$=-1·4367=\bar{2}·5633$, mantissa $=·5633$, characteristic$=-2$.

COMMON LOGARITHMS

14 Number 10 forms a very convenient base for calculations with the help of logarithms, because

(*i*) The characteristic of the logarithm of any number can be determined by inspection.

(*ii*) If the logarithm of any number be known, the logarithm of the multiple or submultiple of that number by any integral power of 10, can be immediately determined. These points are illustrated in Arts. 15 and 16 below.

15. (1)

$$10^0=1$$
$$10^1=10$$
$$10^2=100$$
$$10^3=1000$$

(*a*) All the numbers greater than 1 and having one digit in their integral parts (4·3, 8·1, 9·8, etc.) lie between 10^0 and 10^1, therefore $\log_{10}$ (4·3, 8·1, 9·8, etc.) lie between 0 and 1, *viz.*, it is equal to $0+a$ decimal fraction. Thus the characteristic of the logarithm of a number with one digit in its integral portion is 0.

(*b*) All the numbers having two digits in their integral parts (35·9, 64·32, 93·0, etc.) lie between 10^1 and 10^2, therefore the logarithms of numbers with two integral digits are $1+$.....

Thus the characteristic of the logarithm of a number having 2 digits in its integral part is 1.

In general the logarithm to the base 10, of a number greater than unity and having n digits in its integral parts has $n-1$ as its characteristic.

(2)
$$10^0=1$$
$$10^{-1}=\cdot1$$
$$10^{-2}=\cdot01$$
$$10^{-3}=\cdot001$$

(*a*) All the numbers less than one and having one zero after the decimal point (·02, ·034. etc.), being greater than ·01 and less than ·1 lie between 10^{-2} and 10^{-1}. The logarithms of such numbers to the base 10 lie between -2 and -1. *viz.*, they are equal to $-2+$ some decimal portion (the mantissa being always positive). Thus the characteristic of such numbers is -2.

(*b*) Numbers having two zeros after the decimal point lie between 10^{-3} and 10^{-2}, therefore the characteristic in the logarithms of numbers having two zeros after the decimal points is -3.

In general the logarithm to the base 10 of a decimal fraction having n zeros immediately after the decimal point has $-(n+1)$ as its characteristic.

Logarithms to the base 10 are called **Common Lagarithms** and the base is generally omitted in writing. Log_{10} N is simply written as log N.

Example 1. Find the characteristic in log 8434·23.

The number has got 4 digits in its integral portion,

$\therefore$ characteristic$=3$,

Example 2. What is the characteristic in log 0·038 ?

There is one zero after the decimal point.

$\therefore$ characteristic$=-2$.

16. Consider numbers like 3·85, 385 and 0·0385.

Let the logarithm of 3·85 be x.

(**x**) is only a decimal fraction because there being only one digit in the integral part of 3·85, the characteristic is 0.)

$$\begin{aligned} (a)\quad \log 385 &= \log(3{\cdot}85 \times 10^2) \\ &= \log 3{\cdot}85 + \log 10^2 \\ &= \log 3{\cdot}85 + 2 \log 10 \\ &= x + 2 \qquad [\because \ \log_{10} 10 = 1] \end{aligned}$$

$$\begin{aligned} (b)\quad \log 0{\cdot}0385 &= \log\left(3{\cdot}85 \times \tfrac{1}{100}\right) \\ &= \log\ (3{\cdot}85 \times 10^{-2}) \\ &= \log 3{\cdot}85 + \log 10^{-2} \\ &= \log 3{\cdot}85 - 2 \log 10 \\ &= x - 2. \end{aligned}$$

Numbers 3·85, 385, and 0·0385 have the same significant digits and differ in the position of the decimal point only, and their logarithms x, $2+x$, $-2+x$ have the same mantissa **(x)** and differ in characteristic only. Thus whatever the position of the decimal point, numbers having the same significant digits have the same mantissa in their logarithms.

The multiplication or division by 10 or its any integral power shifts the position of the decimal point only and does not change the order of the significant digits, therefore if the logarithm of any number is given, the logarithm of its multiple or submultiple by an integral power of 10 is found by changing the characteristic to its proper value and keeping the mantissa the same.

Example 1. Given log 48·9 = 1·6893, find log 0·489.

The significant figures in 48·9, and 0·489, are the same, therefore the mantissa in their logarithms must be the same. The characteristic in the logarithm of 0·489 is -1, therefore

$$\log 0{\cdot}489 = -1 + 0{\cdot}6893 = \bar{1}{\cdot}6893.$$

Example 2. Given log 7 = ·8451, find the logarithm of

$$(\cdot 007)^{\frac{1}{4}} / (\cdot 07)^3.$$

The significant digits in 7, 0·007 and 0·07 are the same, so the mantissa in their logarithms are the same. Also the characteristic in the logarithms of 0·007 and 0·07 are -3 and -2 respectively. Hence

$$\log 0{\cdot}007 = -3 + {\cdot}8451 = \bar{3}{\cdot}8451$$
$$\log\ 0{\cdot}07 = -2 + {\cdot}8451 = \bar{2}{\cdot}8451.$$

Therefore,

$$\begin{aligned}\log \frac{({\cdot}007)^{\frac{1}{4}}}{({\cdot}07)^3} &= \log\ ({\cdot}007)^{\frac{1}{4}} - \log({\cdot}07)^3 \\ &= \tfrac{1}{4} \log\ ({\cdot}007) - 3 \log\ ({\cdot}07) \\ &= \tfrac{1}{4}\ (\bar{3}{\cdot}8451) - 3\ (\bar{2}{\cdot}8451) \\ &= \tfrac{1}{4}\ (-3 + {\cdot}8451) - 3(-2 + {\cdot}8451) \\ &= -0{\cdot}75 + 0{\cdot}2113 + 6 - 2{\cdot}5353 \\ &= 6{\cdot}2113 - 3{\cdot}2853 \\ &= 2{\cdot}9260.\end{aligned}$$

The mantissae of the common logarithms of various significant digits have once for all been calculated and written in the form of tables for ready reference. The characteristics can be determined by inspection, therefore the logarithm of any number can be found by the help of the tables.

LOGARITHMIC TABLES

17. Tables of logarithms correct up to different places of decimals are available. In the appendix, tables correct up to four places of decimals, and giving the mantissae of numbers up to four digits are given. These tables are known as **four figure tables**. A portion of these tables is given on page 17 as a specimen.

(1) The column on the extreme left hand gives numbers of two digit.

(2) The figures 0, 1, 2,...9, at the top, correspond to the third digit.

(3) Figures 1, 2,......9, under the heading Mean Differences on the right, correspond to the fourth digit.

The mantissae are all decimal fractions but the decimal point has been omitted throughout to avoid repetition.

The following example illustrates the method of finding the logarithm of a given number.

Example. Find log 432·4

The number has three digits to the left of the decimal point, therefore the characteristic is 2.

For mantissa, look for 43 in the extreme left column and find the number in the horizontal line against this 43 and under 2 (the third digit) at the top. This is found to be 6355. In the same horizontal line under the Mean Difference 4 (the fourth digit), a number 4 is obtained.

Adding both these we get ·6359 as the mantissa. Thus log 432·4=2·6359. (Decimal point should always be added to the mantissa because it is a fraction, by the definition.)

The mantissa of all the numbers having the same significant digits in the same, therefore 0·6359 is the mantissa of all the numbers like 4·324, 43·24, 4324, 0·4324, ·04324, etc. The characteristics of these numbers are, 0, 1, 3,—1, and—2, respectively, therefore the logarithms of these numbers are ·6359, 1·6359, 3·6359, 1·6359 and 2·6359, respectively. The position of the decimal point has no effect on the mantissa, therefore it is not cared for while finding the mantissa from the table.

The actual value of 1·6359 is $-1+\cdot6359=-\cdot3641$ and that of 2·6359 is $-2+\cdot6359=-1\cdot3641$ but in calculations with logarithms it is convenient to keep them as 1·6359 and 2·6359 etc. The significance of the negative sign at the top should be clearly understood.

ANTILOGARITHMS

13. If x is equal to log N, N is called the **antilogarithm** of x.

When the logarithm of a number is known, the number, *viz.*, the antilogarithm, can also be found from the logarithmic tables. The following example illustrates the method.

Example. Find the number whose logarithm is 1·6195.

In the table on page 17 find the position of the number nearest to 6195 but less than it. We have 6191 against 41 and under 6, therefore the first three figures are 416.

There is yet a difference of 4. Look at the column of Mean Difference in the same horizontal line. A difference of 4 corresponds to figure 4 at the top therefore the fourth digit of the number is 4.

Thus the significant digits of the number are 4164. The characteristic in the given logarithm is 1, therefore there are two figures to the left of the decimal point in the required number.

Thus the number is 41·64.

[**Note**. In finding antilogarithms the characteristic is at first neglected because, as the mantissa depends only on the significant digits, the significant digits depend only on the mantissa. The use of the characteristic is simply to determine the position of the decimal point in the number.

Separate tables known as tables of antilogarithms are also available and the method of using them is similar to that of logarithmic tables.

USE OF LOGARITHMS

It is easier to find the value of a product, quotient, square, square root, cube root, etc., by the help of logarithms than by the direct processes of arithmetic.

Find the value of $\dfrac{m\times n\times r}{p\times q}$

Let $\dfrac{m\times n\times r}{p\times q}=M$

$$\log M=\log\frac{m\times n\times r}{p\times q}$$

$$=\log m+\log n+\log r-\log p-\log q=x$$

(say).

Find log m, log n, log p, etc., from the tables and add and subtract them to find x.

Having found x, find the number whose logarithm is x, *viz.*, find the antilogarithm of x. This is the required value. The processes of multiplication and division have been replaced by addition and subtraction which are much simpler.

	0	1	2	3	4	5	6	7	8	9	MEAN DIFFERENCES 1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9

19. Find the value of $\sqrt[n]{m}$

Let $$\sqrt[n]{m}=M$$

$$\log M=\log \sqrt[n]{m}=\frac{1}{n}\log m=x \text{ (say).}$$

The value of log m is found from the tables and divided by n : thus x is obtained. Antilog of x gives M.

At first the finding of logarithms may appear more tedious than multiplication and division, but after some practice the use of logarithms becomes very easy and convenient.

Caution. It would be ridiculous to use logarithms in questions which can be solved mentally, *e.g.*, $\frac{10\times 8}{12}$ etc. It is only where the actual process of multiplication or division, etc., is to be done on paper that ogarithms are to be used.

SOLVED EXAMPLES

Example 1. Given log $2=\cdot 3010$, and log $3=\cdot 4771$. Find (*i*) log $\cdot 3\dot{5}$, (*ii*) antilog $\cdot 7781$, (*iii*) antilog $\cdot 1761$.

(*i*) $\cdot 3\dot{5}=\frac{35-3}{90}=\frac{32}{90}=\frac{2^5}{10\times 3^2}$

$$\log \cdot 3\dot{5}=\log\frac{2^5}{10\times 3^2}=\log 2^5-\log 10-\log 3^2$$

$$=5\log 2-1-2\log 3=1\cdot 5050-1-\cdot 9542$$
$$=\bar{1}\cdot 5508.$$

(*ii*) If M is the antilog of $\cdot 7781$.

$$\log M=\cdot 7781=\cdot 3010+\cdot 4771=\log 2+\log 3$$
$$=\log 2\times 3=\log 6$$

$\therefore M=6.$

(*iii*) Let antilog $\cdot 1761=M$.

$$\log M=\cdot 1761=\cdot 4771-\cdot 3010=\log 3-\log 2=\log \tfrac{3}{2}$$

$$\therefore\quad M=\tfrac{3}{2}=1\cdot 5$$

Example 2. Find the value of $\frac{433\times3\cdot91+\sqrt{212}}{9634\times(23)^3}$ by the use of tables.

$$\log\frac{432\times3\cdot91\times\sqrt{212}}{9634\times(23)^3}$$

$$[+]=\log 432+\log 3\cdot91+\tfrac{1}{2}\log 212-\log 9634-3\log 23$$

$$=2\cdot6355+\cdot5922+\tfrac{1}{2}(2\cdot3263)-3\cdot9838-3(1\cdot3617)$$

from the tables. The addition and subtraction can be performed in one process as under :—

$$\begin{array}{r} 2\cdot6355 \\ \cdot5922 \\ 1\cdot1632 \\ \hline 3\cdot9831 \\ 4\cdot0851 \\ \hline 4\cdot3220 \end{array}$$

Antilog 4·3220=·0002099 which is the required value.

Example 3. Find the value of—

$$39\cdot24\times\cdot00456\times9^5-\frac{4340}{38\cdot64}$$

This quantity is a difference of two quantities and there does not exist any logarithmic relation for a difference, therefore, the value of each should first be found separately and then difference obtained. Bare process is shown below. Logs have been found from the tables and written directly below each other, as under :—

(1)	(2)	
·5937	3·6375	1056
3 6590	1·5871	112·3
4·7710	2·0504	943·7
3·0237		
=1056	=112·3	

[+] After some practice these formalities are omitted and we proceed at once to find the logarithm and arrange them one below the other.

The required value=943·7.

Example 4. Given log 6·833=0·8346, log 8·266=0·9173, and log 4·906=·6907 ; find the square root and seventh root of 0·006833.

The significant figures of 0·006833 and 6·833 are the same therefore the mantissae in their logarithms are equal. Characteristic in log of 0·006803 being equal to $\bar{3}$.

$$\log 0\cdot006833=\bar{3}\cdot8346.$$

(*i*) $\log \sqrt[2]{\cdot006833}=\frac{1}{2}\log 0\cdot006833=\frac{1}{2}(\bar{3}\cdot8346)$

$$=\tfrac{1}{2}(\bar{4}+1\cdot8346)=\bar{2}\cdot9173.$$

[$\bar{3}$ is not exactly divisible by 2, and we cannot carry 1 (left over) to the mantissa as 1 is negative and the mantissa is always positive. To remove this difficulty $\bar{3}\cdot8346$ has been written as $\bar{4}+1\cdot8346$ and then divided by 2.]

Antilog $\bar{2}\cdot9173$=0·08266 (the mantissa being ·9173 the significant digits of the number are 8266 as given.)

∴ $\sqrt[2]{\cdot006833}=\cdot08266.$

(*ii*) $\log \sqrt[7]{\cdot006833}=\frac{1}{7}\log \cdot006833=\frac{1}{7}(\bar{3}\cdot8346)$

$$=\tfrac{1}{7}(\bar{7}+4\cdot8346)=\bar{1}\cdot6907$$

Antilog $\bar{1}\cdot6907$=0·4906.

∴ $\sqrt[7]{\cdot006833}=0\cdot4906.$

Note. 4·8346 is not exactly divisible by 7 but we are using four figure logarithmic tables, therefore no process need be carried on to more than four decimal places. The last digit (7) of $\bar{1}\cdot6907$ has been taken as the nearest possible digit.

Example 5. Deduce the relation between the variables l and T from the equation, $\log l=\log k+2\log T$. where k is a constant. What well-known relation does it represent in Physics ?

The equation $\log l=\log k+2\log T$ can be written as

$\log l=\log k+\log T^2$ [∵ $\log m^r+r\log m$]

or $\log l=\log kT^2$ [∵ $\log m\times n=\log m+\log n$]

or $l=kT^2$.

or $$\frac{l}{T^2}=k.$$

If l denotes the length of a simple pendulum and T its period of vibration $\frac{l}{T^2}$ is constant, is the relation expressed by this equation.

EXAMPLES III.

1. Write down the characteristics in the logarithms of :—

(*i*) 3435, (*ii*) ·98, (*iii*) ·0004378, (*iv*) 654217, (*v*) 30000.

2. The logarithm of 3789 is 3·5785. Write down the logarithms of—(*i*) 3·789, (*ii*) 378·9, (*iii*) ·03789, (*iv*) ·3789, (*v*) ·00008789.

3. The logarithm of 2·984 is ·4748. Write down the antilogarithms of—

(*i*) 3·4748, (*ii*) 4·4748, (*iii*) 2·4748, (*iv*) $\bar{1}\cdot4748$.

When required the following logarithms may be used :—

log 2 = ·3010, log 3 = ·4771, log 7 = ·8451.

Find the value of—

4. log 32. **5.** log 35·28. **6.** $\log \sqrt[5]{\cdot00162}$.

7. log 0·217 **8.** $16 \log \frac{10}{9} - 4 \log \frac{25}{24} - 7 \log \frac{80}{81}$.

Use four figure tables wherever necessary in the following questions :—

Evaluate :—

9. $3\cdot7 \times 8\cdot9 \times \cdot023$. **10.** $43 \times 8\cdot07 \times \cdot0392$. **11.** $8\cdot034 \times 1893$

12. $\frac{\cdot2179}{\cdot08973}$. **13.** $\frac{17\cdot3}{294\cdot8}$. **14.** $\sqrt[3]{82\cdot56}$.

15. $\frac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{245}}$. **16.** $(\cdot089)^{\frac{4}{7}}$.

17. $\sqrt[5]{\cdot003456}$. **18.** $\sqrt[5]{\cdot000003456}$.

19. Given log 5·849 = 0·7671 and log 9·145 = ·9612 ; find the value of $(8\cdot549)^{\frac{1}{5}}$.

20. Find the value of $\sqrt[5]{0{\cdot}00000165}$. Given log 165=2·2175, log 6974=3·8435. (*P.U. 1926*)

21. Log C=log k+log tan θ, represents in logarithmic form a well-known relation in physics. Transform it to the ordinary form and explain what it is.

22. Deduce the relation between the variable L and P from the equation log L=Log K+2 Log P, where K is a constant, and show that the relation represents a well-known phenomenon in Physics. (*P.U. 1929*)

23. Given that log 8·1=0·9085 and log 6·05=0·7817 ; find the fifth root of 0·81. (*P.U. 1930*)

24. What root is ·02090 of ·0004367 ?

25. Time of vibration of a simple pendulum is given by the formula $T=2\pi\sqrt{\dfrac{l}{g}}$ where l is the length of the pendulum and $g=981$, is the acceleration of gravity. Find the time of vibration of a simple pendulum having a length equal to 80 cm.

26. Using the formula of the last question, find the length of the simple pendulum which makes one vibration in 2 seconds (Second's pendulum.)

27. Frequency of vibration of a stretched string is given by the formula $n=\dfrac{l}{2l}\sqrt{\dfrac{T}{m}}$ where l is the length of the string, T the stretching force and m the mass per unit length of the string. Find the frequency of a string 100 cm. long stretched by a force of 25,000,000 dynes and weighing ·024 gm. per centimetre.

CHAPTER IV

QUADRATIC EQUATIONS

20. An equation always contains certain unknown quantities and certain other known ones.

The solution of an equation means the finding of suitable values for the unknown quantities, so that the relation expressed by the equation is satisfied when these values are submitted. These values are called the *roots* of the equation.

Unknown quantities are generally denoted by the last letters of the alphabet x, y, z, etc., and the known, by the first letters, a, b, c, etc.

The student is already familiar with equations of the first degree, *i. e.*, those which contain terms of the first and not of any higher power of the unknown quantities.

21. A Quadratic Equation contains terms involving second power of the unknown quantities and none of a power higher than the second, *e.g.*,

1. $14x-x^2=33.$
2. $x^2-4=\frac{1}{3}(x-5).$
3. $x^2+y^2+xy+2x+3y=0.$

Quadratic equation containing one unknown quantity only will be dealt with in this book.

22. To Solve the Quadratic $ax^2+bx+c=0$.

After simplification and proper arrangement of terms every quadratic equation can be expressed in the form

$$ax^2+bx+c=0 \qquad \ldots (1)$$

where a, b, c are any known quantities or constants, and a is not zero, for if $a=0$, the equation does not remain a quadratic.

To solve (1) divide it by a, then

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Transpose $\frac{c}{a}$ to the right-hand side, then

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Complete the square on the left-hand side by adding $\frac{b^2}{4a^2}$ to both sides ; we have

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}, \text{ or } \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Taking square-roots on both sides, we obtain

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence $$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

are the two roots of the given quadratic.

23. Any quadratic equation after it has been brought into the general form can be solved by simply substituting the values of a, b and c in the formula given above, which is important and must be remembered.

Example 1. Solve $3x^2-4x-4=0$.

In this equation $a=3$, $b=-4$, and $c=-4$.

$$\therefore\ x=\frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-4)}}{2\times3}$$

$$=\frac{4\pm\sqrt{16+48}}{6}=\frac{4\pm8}{6}=2 \text{ or} -\frac{2}{3}$$

Example 2. Solve $x(2x-5)=2(2x-4)$.

$$2x^2-5x=4x-8$$

i.e., $$2x^2-9x+8=0$$

$$\therefore\ x=\frac{9\pm\sqrt{81-64}}{4}=\frac{9\pm\sqrt{17}}{4}$$

$$=\frac{9+\sqrt{17}}{4} \text{ or } \frac{9-\sqrt{17}}{4}.$$

24. Sometimes it is easier to solve a quadratic equation by factorising it.

Example 3. Solve $x^2-15x+56=0$.

Factorising, we have $(x-7)(x-8)=0$.

hence either $x-7=0$

or $$x-8=0$$

$$\therefore\quad x=7 \text{ or } 8.$$

This method is only useful when the factors are apparent.

SOLVED EXAMPLES

Example 1. Solve

$$10(2x+3)(x-3)-(7x+3)^2+22(x+3)(x-1)+200=0$$

$$10(2x^2-3x-9)-(49x^2+42x+9)+22(x^2+2x-3)+200=0$$

$$\therefore\quad -7x^2-28x+35=0 \quad\text{or}\quad x^2+4x-5=0$$

i.e., $$(x+5)(x-1)=0 \quad\text{or}\quad x=-5 \text{ or } +1.$$

Example 2. Solve $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

$$\frac{1}{a+b+x}=\frac{ab+ax+bx}{abx}$$

Multiplying crosswise,

$$abx=a^2b+a^2x+abx+ab^2+abx+b^2x+abx+ax^2+bx^2.$$

Transposing and arranging the terms in descending powers of x.

$$x^2(a+b)+x(a^2+b^2+2ab)+a^2b+ab^2=0$$

$$\therefore\quad x^2(a+b)+x(a+b)^2+ab(a+b)=0.$$

[†]Dividing all the terms by $(a+b)$ which does not involve x.

$$x^2+x(a+b)+ab=0$$

$$\therefore\quad (x+a)(x+b)=0.$$

i.e., $$x=-a \text{ or } -b.$$

[†]**Note.** When dividing all the terms of an equation by a common factor see if the factor contains the unknown quantity (x) or not. If it contains x, one value of x can be obtained by equating the factor to zero as illustrated in the method of solving an equation by factorization.

EXAMPLES IV

Solve by factorization—

1. $x^2-5x+4=0.$ **2.** $x^2+x-2=0.$ **3.** $x^2-x-6=0.$

4. $x^2+17x+70=0$. **5.** $x^2+34x=336$.

Solve—

6. $6x^2-13x+6=0$. **7.** $12x^2-7x-12=0$.

8. $7x^2-3x=160$. **9.** $780x^2-73x+1=0$.

10. $4(x^2-1)=4x-1$. **11.** $(3x-5)(2x-5)=(x+3)(x-1)$.

12. $(x+1)(2x+3)=4x^2-22$.

13. $(x-1)(x-2)+(x-2)(x-4)=6(2x-5)$.

14. $\frac{5}{7}x^2+\frac{7}{5}x+\frac{73}{140}=0$. **15.** $\frac{x}{2}+\frac{2}{x}=\frac{x}{3}+\frac{3}{x}$.

16. $\frac{x}{7}+\frac{21}{x+5}=\frac{23}{7}$. **17.** $\frac{x}{x-1}=\frac{3}{2}+\frac{x-1}{x}$.

18. $\frac{x-6}{x-12}-\frac{x-12}{x-6}=\frac{5}{6}$. **19.** $\frac{5}{x+2}+\frac{3}{x}=\frac{14}{x+4}$.

20. $\frac{3x-2}{2x-5}-\frac{2x-5}{3x-2}=\frac{8}{3}$. **21.** $\frac{x-2}{x+2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$.

22. $10(2x+3)(x-3)+(7x+3)^2=20(x+3)(x-1)$.

23. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}$. **24.** $\frac{1}{x+a}+\frac{1}{x+2a}+\frac{1}{x+3a}=x$.

25. $\frac{1}{(x-b)(x-c)}+\frac{1}{(a+b)(a+c)}=\frac{1}{(a+c)(x-c)}+\frac{1}{(a+b)(x-b)}$.

26. $(a-b)x^2-(a+b)x+2b=0$.

27. $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$. **28.** $(ax-b)(bx-a)=c$.

CHAPTER V

MISCELLANEOUS EQUATIONS SOLVED LIKE QUADRATIC EQUATIONS

25. Certain types of equations which are not quadratic can be solved by reducing them to the quadratic form. In general, each type has its own method of solution.

Example 1. $x^{\frac{1}{n}} - 5x^{-\frac{1}{n}} + 4 = 0.$

$$\text{Put } x^{\frac{1}{n}} = y.$$

$$\therefore\ x^{-\frac{1}{n}} = y^{-1} = \frac{1}{y}$$

The equation becomes $y - \frac{5}{y} + 4 = 0.$

$$\therefore\quad y^2 + 4y - 5 = 0.$$

i.e., $$(y+5)(y-1) = 0.$$

$$y = -5 \text{ or } 1.$$

$$\therefore\quad x^{\frac{1}{n}} = -5 \text{ or } 1.$$

or $$x = (-5)^n \text{ or } 1^n (=1).$$

Example 2. $15\sqrt{\frac{5}{x}} + 16\sqrt{\frac{x}{5}} - 32 = 0.$

Put $\sqrt{\frac{x}{5}} = y$, then $\sqrt{\frac{5}{x}} = \frac{1}{y}$

$$\frac{15}{y} + 16y - 32 = 0$$

$$16y^2 - 32y + 15 = 0$$

$$y = \frac{32 \pm \sqrt{1024 - 960}}{32}$$

$$= \frac{32 \pm 8}{22} = \frac{5}{4} \text{ or } \frac{3}{4}$$

i.e., $$\sqrt{\frac{x}{5}} = \frac{5}{4} \text{ or } \frac{3}{4}$$

i.e., $$\frac{x}{5} = \frac{25}{16} \text{ or } \frac{9}{16}$$

i.e., $$x = -\frac{13}{16} \text{ or } 2\frac{13}{16}$$

Example 3. $\sqrt{3x^2 + 5x + 7} - \sqrt{3x^2 + 5x - 6} = 1$ (*P. U. 1944*)

Put $\sqrt{3x^2 + 5x - 6} = y.$

$$\therefore \quad \sqrt{3x^2+5x+7}=\sqrt{y^2+13}.$$

The equation becomes $\sqrt{y^2+13}-y=1$

$$\therefore \quad \sqrt{y^2+13}=1+y$$

Squaring both sides $y^2+13=(1+y)^2$

$$=1+2y+y^2$$

$$\therefore \quad 2y=13-1=12$$

or $y=6$

$$\therefore \quad \sqrt{3x^2+5x-6}=6$$

or $3x^2+5x-6=6^2=36$

or $3x^2+5x+42=0$

$$x=\frac{-5\pm\sqrt{25+504}}{6}$$

$$=\frac{-5\pm23}{6}=3 \text{ or } -4\tfrac{2}{3}.$$

Example 4. $3x^2+15x-2=2\sqrt{x^2+5x+1}$

$3(x^2+5x+1)-5=2\sqrt{x^2+5x+1}$ [note this step].

Put $\sqrt{x^2+5x+1}=y$

The equation becomes $3y^2-5=2y$

or $3y^2-2y-5=0$

or $(y+1)(3y-5)=0$

or $y=-1, \text{ or } \frac{5}{3}.$

$$\therefore \quad \sqrt{x^2+5x+1}=-1 \text{ or } \tfrac{5}{3}.$$

or $x^2+5x+1=1 \quad \text{or} \quad \frac{25}{9}.$

Taking first value

$$x^2+5x+1=1$$

or $x^2+5x=0$

or $x(x+5)=0$ or $x=0$ or -5 ;

Taking second value

$$x^2+5x+1=\tfrac{25}{9}$$

or $$x^2+5x-\frac{16}{9}=0$$

$$9x^2+45x-16=0$$

$$9x^2+48x-3x-16=0$$

$$3x(3x+16)-(3x+16)=0$$

or $$(3x+16)(3x-1)=0, \text{ or } x=-5\tfrac{1}{3} \text{ or } \tfrac{1}{3}.$$

Example 5. Solve $x+y=10.........(i)$

$$xy=\frac{375}{16} ......(ii)$$

$$(x-y)^2=(x+y)^2-4xy=100-\frac{375}{4}=\frac{25}{4}$$

$$\therefore \quad x-y=\pm\tfrac{5}{2}......(iii)$$

Taking the upper sign of (*iii*) and adding (*i*) and (*iii*) we get

$$2x=\tfrac{25}{2}\ ; \qquad \therefore \quad x=\tfrac{25}{4}.$$

Subtracting (*iii*) from (*i*) $2y=\tfrac{15}{2}$; $\therefore$ $y=\tfrac{15}{4}$.

Taking the lower sign of (*iii*) and adding (*i*) and (*iii*) we get

$$2x=\tfrac{15}{2}\ ; \qquad \therefore \quad x=\tfrac{15}{4}$$

Subtracting (*iii*) from (*i*) $2y=\tfrac{25}{2}$; $\therefore$ $y=\tfrac{25}{4}$.

It will be noted that with the negative sign of $\sqrt{\tfrac{25}{4}}$ the values are the same as with the positive sign but have simply been interchanged, therefore only positive sign is taken in such questions.

Alternative Method.

$$y=\frac{375}{16x} \text{ from } (ii).$$

$$\therefore \quad x+\frac{375}{16x}=10 \text{ by substitution in } (i)$$

i e., $16x^2-160x+375=0$

$$x=\frac{160\pm\sqrt{25600-24000}}{32}=\frac{160\pm40}{32}=\frac{25}{4} \text{ or } \frac{15}{4}.$$

Similarly $y=\tfrac{15}{4}$ or $\tfrac{25}{4}$.

Example 6. $x+y=13.....(i)$; $xy=30.........(ii)$

In simple cases like this the values can be easily guessed as for factorization.

Thus $x=10$ and $y=3$ or $y=10$ and $x=3$.

The very forms of the equations show that x and y are interchangeable.

EXAMPLES V

Solve—

1. $8x^{\frac{3}{2n}}-8x^{-\frac{3}{2n}}=63$.
2. $6x^{\frac{1}{2}}=5x^{-\frac{1}{2}}-13$.
3. $x^6-35x^3+216=0$.
4. $\dfrac{x^2-1}{3}+\dfrac{1}{x^2}=1$.
5. $x+5-\sqrt{x+5}=6$.
6. $x^{\frac{1}{3}}+\dfrac{5}{2x^{\frac{1}{3}}}=\dfrac{13}{4}$.
7. $x+y=20$; $xy=75$.
8. $x+y=\frac{3}{2}$; $xy=\frac{1}{2}$.
9. $x+y=85$; $xy=474$.
10. $x-y=15$; $xy=126$.
11. $x-y=3$; $xy=154$.

12. The length of a rectangular field exceeds the breadth by one yard and the area of the field is 14,520 sq. yards. Find the lengths of the sides.

13. Find the number whose square added to its cube is nine times the next higher number.

14. A person bought a certain number of oxen for £80; if he had bought 4 more for the same sum each ox would have cost £1 less. Find the number of oxen and the price of each.

CHAPTER VI

NATURE OF ROOTS, ETC.

26. Real and Imaginary Quantities. All positive and negative quantities integral or fractional are called **real quantities**, but the square root of a negative quantity cannot be real because the squares of all real quantities, positive as well as negative, are positive. There is no real quantity whose square may be negative, therefore the square root of a negative quantity is called an **imaginary quantity**. $\sqrt{-1}$ is a typical imaginary quantity because every imaginary quantity can be put in the form of the product of a real quantity and $\sqrt{-1}$. Thus $\sqrt{-25}$ can be written as $5\sqrt{-1}$.

27. Roots of a quadratic equation have previously been found to be $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

(*i*) If $b^2>4ac$; b^2-4ac is positive, therefore the roots are real and unequal.

(*ii*) If $b^2=4ac$; $b^2-4ac=0$, therefore the roots are real and equal.

(*iii*) If $b^2<4ac$; b^2-4ac is negative and $\sqrt{b^2-4ac}$ is imaginary, therefore both the roots are imaginary.

Example 1. Find the nature of the roots of the equation

$$3x^2+4x+2=0.$$

b^2-4ac of this equation $=16-24=-8$. This is negative, therefore the roots are imaginary.

(The actual values of the roots are $\frac{-4\pm\sqrt{-8}}{6}$).

Example 2. Find the nature of the roots of the equation

$$12x^2+29x+15=0.$$

b^2-4ac of this equation is $841-720=121$.

This is positive, therefore the roots are real and different.

(The actual values are $\frac{-3}{4}$ and $\frac{-5}{3}$ as found by solving the equation.)

Example 3. Distance travelled by a body starting with a velocity u and moving with uniform acceleration a, in a time t, is given by the formula $s=ut+\frac{1}{2}at^2$. Find the time taken by a body starting with a velocity of 2 ft. per second and moving with an acceleration of 4 ft. per second per second, to cover a distance of 40 ft.

In the given data $u=2$; $a=4$; and $s=40$. Substituting these values in the given formula we get $40=2t+\frac{1}{2}\times4\times t^2$,

$$\text{or} \quad 2t^2+2t-40=0$$

i. e., $t^2+t-20=0$

i. e., $(t+5)(t-4)=0$;

$\therefore \quad t=-5$ or 4.

Negative value of t is absurd because no distance could have been moved before starting. This value should, therefore, be rejected and the answer is 4.

Example 4. Employing the formula of the last example, find after how much time will a body be at a distance of 30 ft. from its starting point, if it starts with a velocity of 11 ft. per second and moves with an acceleration of -2 ft. per second per second.

$$u=11\ ;\ a=-2\ ;\ s=30$$

$$s=ut+\tfrac{1}{2}at^2$$

$$\therefore \quad 30=11t+\tfrac{1}{2}(-2)t^2=11t-t^2$$

$$i.\ e., \quad t^2-11t+30=0$$

$$(t-5)(t-6)=0$$

$$\therefore \quad t=5 \text{ or } 6.$$

Both values of t are positive, therefore both must be admissible.

Explanation. The acceleration is negative which means that the velocity is decreasing. In 5 seconds the body travels a distance of 30 ft and moves on with gradually decreasing velocity till it comes to rest when the velocity becomes zero. As the negative acceleration continues, it begins to move backward and once again in its backwards journey reaches the place, 30 ft. away from its position of start after 6 seconds.

Note. Bodies moving freely under the action of gravity move with a downward acceleration of 32 ft. per second per second and the formula for the distance travelled is the same as in Example 3.

28. In the solution of a quadratic equation by the method of factorisation it was observed that the roots are obtained by equating each of the factors to zero.

Conversely if α and β are the roots of an equation ;

$$x=\alpha \text{ and } x=\beta \text{ or } x-\alpha=0, \text{ and } x-\beta=0.$$

Thus $(x-\alpha)(x-\beta)=0$ will evidently be the equation whose roots are α and β.

Example 5. Form an equation whose roots are 4 and -3. Factors of the equation are $x-4$ and $x+3$.

$\therefore$ the required equation is $(x-4)(x+3)=0$

or $$x^2-x-12=0$$

Example 6. The sum of the roots of an equation is 8 and the sum of their reciprocals is $\frac{8}{15}$. Find the equation.

Let α and β be the roots

$$\alpha+\beta=8 \qquad \ldots \quad \text{(i)}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{8}{15} \qquad \ldots \quad \text{(ii)}$$

Equation (ii) becomes $\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{8}{15}$.

Substituting the value of $\alpha+\beta$ from (i) we have

$$\frac{8}{\alpha\beta}=\frac{8}{15}$$

$$\therefore \qquad \alpha\beta=15,$$

The equation with roots α and β is

$$(x-\alpha)(x-\beta)=0$$

i.e. $$x^2-(\alpha+\beta)x+\alpha\beta=0 \qquad \ldots\text{(iii)}$$

Substituting the values of $\alpha+\beta$ and $\alpha\beta$ found above, (iii) becomes

$$x^2-8x+15=0.$$

This is the required equation.

Note. If the sum and the product of the roots is known the equation is easily formed as shown above without finding the value of the roots.

MAXIMUM AND MINIMUM VALUES OF AN EXPRESSION IN SOME SIMPLE CASES

29. (*i*) If an expression can be put in the form $a+(b+x)^2$, the second term is a perfect square and therefore always positive for all *real* values of x. The least value it can have is zero, therefore the least value of $a+(b+x)^2$ is a and this occurs when $b+x=0$ or $x=-b$.

(*ii*) Similarly if an expression can be put in the form $a-(b+x)^2$, its maximum value is a and this occurs when $x=-b$.

Example 7. Find the minimum value of $x^2-8x+21$.

$$x^2-8x+21=(x-4)^2+5.$$

5 is a constant and the least value of $(x-4)^2=0$ when $x=4$, therefore the least value of the expression is 5.

Example 8. For what value of x is $4x-x^2+10$ maximum ?

$$4x-x^2+10=-(x^2-4x-10)$$
$$=-\{(x-2)^2-14\}=14-(x-2)^2$$

The expression has its maximum value when $(x-2)^2$ is minimum. The least value of $(x-2)^2$ is zero when $x=2$.

∴ the expression $4x-x^2+10$ is maximum when $x=2$.

30. ab is constant. Show that $a+b$ will be minimum when $a=b$.

$$a+b=(\sqrt{a}-\sqrt{b})^2+2\sqrt{ab}.$$

Second term is constant, therefore $a+b$ is minimum for minimum value of $(\sqrt{a}-\sqrt{b})^2$ which is zero when $a=b$.

∴ $a+b$ is minimum when $a=b$.

Verify by putting $ab=64$ (say).

Some of the values of a and b are : 2 and 32 ; 4 and 16 ; 8 and 8. The sum of these is minimum when they are equal.

EXAMPLES VI.

Find the nature of the roots of

1. $6x^2+15x+4=0$. **2.** $4x^2+12x+9=0$.

3. $9x^2-10x+1=0$. **4.** $x^2-\frac{2}{3}x+\frac{4}{9}=0$.

5. Write down the roots of the quadratic $ax^2+bx+c=0$, and find the condition that the roots may be (*i*) equal, (*ii*) imaginary. (*P. U. 1926, 1939*)

Form the equations whose roots are

6. 3 and $\frac{1}{3}$. **7.** $\frac{2}{5}$ and $\frac{3}{5}$.

8. $2\pm\sqrt{3}$. **9.** $1\pm\sqrt{2}$.

The sum of the roots and the sum of their reciprocals are given below. Find the equations.

10. -1 and $\frac{1}{6}$. **11.** 7 and $\frac{7}{12}$.

12. 15 and $\frac{5}{12}$. **13.** 25 and $\frac{1}{4}$.

Find the maximum value of

14. $4-x^2+8x$. **15.** $9-4x^2+12x$.

16. $3x-x^2+\frac{11}{4}$. **17.** $6x-9x^2+1$.

Find the minimum value of

18. x^2+2x+9. **19.** $4x^2-16x+20$.

20. $x^2-6x+13$. **21.** $2x^2+x+4$.

22. Given : mn, E, R and r are constant. Show that

$$\frac{nE}{R+\dfrac{nr}{m}} \text{ is maximum when } R=\frac{nr}{m}.$$

23. How much time will a body starting with a velocity of 80 ft. per second and an acceleration of 10 ft. per second per second take in reaching a point 200 feet away from its position of start ?

24. A body is projected vertically upwards with a velocity of 96 ft. per second. After how much time will it return to its point of projection ?

25. A stone is let fall at a height of 100 ft. above the ground from a balloon rising upwards with a velocity of 50 ft per second. After how much time will it (*a*) be 120 ft. from the ground, (*b*) reach the ground ?

26. A stone is thrown up with a velocity of 128 ft. per second, show that it cannot reach a height of 100 yds.

(**Hint** :—Show that the roots are imaginary.)

27. After how much time will a body thrown upwards with a velocity of 160 ft. per second reach a height of 400 ft. ? What is the meaning of only one answer ?

28. A cricket ball is thrown vertically upwards with a velocity of 60 ft. per second. After what time will it be at a height of 50 ft. above the point of projection ? (*P. U. 1926*)

29. A body is projected vertically upwards with a velocity of 112 ft. per second. When will it be at a height of 160 ft. ? ($g=32$)

(*P. U. 1930*)

CHAPTER VII

ARITHMETICAL PROGRESSION

31. Def. Any number of quantities are said to be in **Arithmetical Progression** when the difference of each quantity from the preceding is the same throughout. The various quantities are called the *terms* of the series and the constant difference is called the *common difference* of the series.

The letters, A. P., are used as an abbreviation for the words 'arithmetical progression'.

Thus each of the following series is an Arithmetical Progression :

1. 2, 4, 6, 8, 10,..............
2. 1, 5, 9, 13, 17,..............
3. 20, 17, 14, 11, 8,..............
4. $a, a+d, a+2d, a+3d,$...........

In 1 each term is greater than the preceding one by 2.
,, 2 ,, ,, ,, ,, ,, ,, ,, ,, by 4.
,, 3 ,, ,, ,, less ,, ,, ,, ,, by 3.
,, 4 ,, ,, ,, greater ,, ,, ,, ,, by d.

The common difference is found by subtracting any term from the one that follows it.

32. *To find the nth term of an A. P.*

Let a denote the first term and d the common difference of an A.P. where a and d may have any value. Call the first term as T_1, the second term as T_2, the third term as T_3, and in general the nth term as T_n where n is an integer. The A. P. with a as first term and d as common difference is.

$a, a+d, a+2d, a+3d,$.........
T_1, the first term, is a,
T_2, the 2nd term, is $a+d$.
T_3, the 3rd term, is $a+2d$.
T_4, the 4th term, is $a+3d$.
....
T_{10}, the 10th term, is $a+9d$.

We find that the co-efficient of d in each term is one less than the suffix of T, *viz.*, the number of the term.

Thus $$T_n = a+(n-1)d.$$

Example 1. Find the 20th term of the series,

$$2,\ 6,\ 10,\ldots\ldots$$

$$T_{20} = a+19d = 2+19\times 4 = 78.$$

33. *To find the sum of n terms of an A. P.*

If S_n denotes the sum of n terms and l the last term, *viz.*, the nth term of an A.P. whose first term is a and common difference d.

$$S_n = a+(a+d)+(a+2d)+\ldots\ldots+(l-2d)+(l-d)+l \ldots\ldots \quad (i)$$

Writing the terms in the reverse order,

$$S_n = l+(l-d)+(l-2d)+\ldots\ldots+(a+2d)+(a+d)+a \ldots\ldots \quad (ii)$$

Adding (*i*) and (*ii*) we get

$$2S_n = (a+l)+(a+l)+(a+l)+\ldots\ldots n \text{ terms}$$
$$= n(a+l)$$

$$\therefore\ S_n = \frac{n}{2}(a+l)$$

l is the nth term, which is $= a+(n-1)d$.

$$\therefore\ S_n = \frac{n}{2}\left\{a+a+(n-1)d\right\} = \frac{n}{2}\left\{2a+(n-1)d\right\}$$

Example 2. Find the sum of 30 terms of the series

$$\tfrac{1}{2},\ \tfrac{3}{4},\ 1 \ldots\ldots\ldots\ldots\ldots$$

1st term $(a) = \frac{1}{2}$; Com. diff. $(d) = \frac{1}{4}$; $n = 30$.

$$S_{30} = \tfrac{30}{2}\{2\times\tfrac{1}{2}+(3-1)\times\tfrac{1}{4}\}$$
$$= 15(1+\tfrac{29}{4}) = 15\times\tfrac{33}{4} = \tfrac{495}{4} = 123\tfrac{3}{4}.$$

34. When three quantities are in A. P. the middle one is said to be the **Arithmetic Mean** or simply the mean of the other two.

Let m be the arithmetic mean between a and b. Then a, m, b; form an A. P. (Def.)

$$m-a = b-m$$

$$\therefore \qquad 2m=a+b$$

$$i.e., \qquad m=\frac{a+b}{2}.$$

Thus the arithmetic mean between two quantities is equal to half their sum.

35. Def. When any number of quantities form an A.P., all the quantities except the first and the last are called **Arithmetic** Means between the first and the last quantities.

To insert n arithmetic means between a and b.

Let, $m_1, m_2 \ldots m_n$ be the required means

$a, m_1, m_2 \ldots m_n, b$, form an A, P. with $n+2$ terms ; a is the first term and b the $(n+2)$th term.

If d be the common difference of this series

$$b=T_{n+2}=a+(n+1)d$$

$$\therefore \qquad (n+1)d=b-a.$$

$$\text{or} \qquad d=\frac{b-a}{n+1}$$

$$m_1=a+d=a+\frac{b-a}{n+1}$$

$$m_2=a+2d=a+2\frac{b-a}{n+1} \text{ and so on.}$$

Example 3. Insert 6 arithmetic means between 1 and 29.

Let $m_1, m_2, \ldots\ldots\ldots$ be the means,

then $1, m_1, m_2, \ldots\ldots 29$ form an A. P. with 8 terms.

$$29=1+7d$$

$$\therefore \qquad d=4.$$

Thus the means are 5, 9, 13, 17, 21 and 25.

SOLVED EXAMPLES

Example 4. Find the sum of p terms of a series whose nth term is $3n-2$.

T_n is given$=3n-2$, therefore by giving different values to n beginning from 1, we have,

$$T_1=3\times1-2=1$$

$$T_2=3\times2-2=4$$

$$T_3 = 3\times 3 - 2 = 7$$
$$T_4 = 3\times 4 - 2 = 10$$

Thus the series is 1, 4, 7, 10,............

$$S_p = \frac{p}{2}\left\{2\times 1 + (p-1)3\right\} = \frac{p(3p-1)}{2}.$$

Example 5. Find the A. P. whose 4th term is 9 and 10th term is 21.

Let a be the first term, and d the common difference,

$$a+3d=9\text{...........}(i)$$
$$a+9d=21\text{.........}(ii)$$

Subtracting (i) from (ii), we have

$$6d=12, \qquad \therefore\ d=2.$$
$$\therefore \text{ from } (i) \text{ or } (ii)\ \ a=3.$$

$\therefore$ the series is 3, 5, 7.........................

Example 6. The sum of 15 terms of an A. P. is 600 and the common difference is 5. Find the series,

$$S_{15} = \tfrac{15}{2}\{2a+(15-1)5\} = 600$$

$\therefore$ $$15(a+35)=600.$$

i.e., $$15a = 600-525 = 75$$

i.e., $$a=5.$$

Hence the series is 5, 10, 15..............

Example 7. How many terms of the series 4, 6, 8, 10,...... will amount to 40 ?

Let n be the number of terms required.

$$\frac{n}{2}\left\{8+(n-1)2\right\}=40$$
$$\therefore \quad 4n+n^2-n=40$$
$$n^2+3n-40=0$$
$$\therefore \quad n=\frac{-3\pm\sqrt{9+160}}{2}=\frac{-3\pm 13}{2}=5 \text{ or } -8.$$

We reject -8 because the negative value of n has no meaning. Thus the answer is 5.

Example 8. Show that when a body moves with uniform acceleration in a straight line, the velocities at the ends of successive seconds are in arithmetical progression. (*P.U. 1939*)

Let u feet per second be the initial velocity of the moving body and let a feet per second per second be the uniform acceleration, so that in each successive second the velocity of the body is increased by a feet per second. Thus

Initial velocity $= u$ ft./sec.

velocity at end of first second $= u+a$ ft./sec.

" " " " second " $= u+2a$ ft./sec.

" " " " third " $= u+3a$ ft./sec.

and so on, the velocity at the end of the tth second

$= u+ta$ ft./sec.

All these velocities at ends of successive seconds form an A. P., whose first term is u and the common difference is a.

EXAMPLES VII

Find the

1. 10th term of the series 4, 8, 12, 16,.........
2. 21st term " " " 9, 8, 7, 6, 5,.........
3. 12th " " " " $\frac{1}{3}, \frac{2}{3}, 1$,...............
4. rth " " " " 9, 11, 13,...........

Sum the following :—

5. 116, 108, 100,..................20 terms.
6. $\frac{3}{4}, \frac{2}{3}, \frac{7}{12}$,........................ 10 terms.
7. $2\frac{1}{12}, 2, 1\frac{11}{12}$,......................19 terms.
8. $\frac{8}{5}, \frac{6}{5}, \frac{4}{5}$,............................15 terms.
9. $2, 3\frac{1}{4}, 4\frac{1}{2}$,..............................25 terms
10. $\frac{1}{1+\sqrt{m}}, \frac{1}{1-m}, \frac{1}{1-\sqrt{m}}$, ...$n$ terms.
11. 1·3, 3·1, 4·9 ....10 terms.

12. $\frac{6}{\sqrt{3}}$, $3\sqrt{3}$, $\frac{12}{\sqrt{3}}$ ....40 terms.

13. $x-3y$, $2x-5y$, $3x-7y$, ...n terms.

14. Insert 2 arithmetic means between 1 and 7.

15. ,, 19 ,, ,, ,, $\frac{1}{4}$ and $\frac{-39}{4}$.

16. ,, 6 ,, ,, ,, 5 and $\frac{1}{3}$.

17. Second term of an A. P. is 31 and 7th term is 86. Find the sum of 16 terms.

18. Third term of an A. P. is 36, and 11th term is 132. Find the series.

How many terms of the series

19. $-2, 2, 6,$......................will amount to 160 ?

20. 12, 16, 20......................,, ,, ,, 208 ?

21. 3·75, 3·5, 3·25,................,, ,, ,, 30 ?

22. $\frac{3}{\sqrt{5}}$, $\frac{4}{\sqrt{5}}$, $\sqrt{5}$, ...,, ,, to $75\sqrt{5}$?

23. The first term of an A. P. is 2 and the fifth term is 7. How many terms must be taken to make 204 ?

24. The sum of 3 numbers in A. P. is 30 and their product is 840. Find them.

[**Hint** : Take the numbers as $a-d$, a and $a+d$ and find the value of a by adding them.]

25. If the sum of 7 terms of an A. P. is 49 and the sum of 17 terms is 289, find the sum of n terms.

26. Form an A. P. such that the sum of the first five terms is $\frac{1}{4}$th sum of the following five terms, the first term being 1.

27. The sum of four numbers in A. P. is 20, and the product of the first and 4th is to the product of the 2nd and 3rd as 2 : 3. Find them.

28. Find three numbers in A. P. whose sum is equal to 30 and the least number is $\frac{1}{3}$rd of the greatest.

29. If the sum of r terms of an A. P. is $2r+3r^2$, find the pth term.

[**Hint** : pth term is equal to the sum of p terms minus the sum of $p-1$ terms$=2p+3p^2-2(p-1)-3(p-1)^2$.]

30. The first term of an A. P. is one and the sum of n terms is n^2; find the common difference.

31. Prove that the squares of x^2-2x-1, x^2+1 and x^2+2x-1 are in A. P.

32. If $a, b, c, d,$............are in A.P., show that the following series are also in A. P.

(*i*) $a+x, b+x, c+x,$...........

(*ii*) $a-x, b-x, c-x,$...........

(*iii*) $ax, bx, cx,$.....................

(*iv*) $\frac{a}{x}, \frac{b}{x}, \frac{c}{x},$ ....

33. The interior angles of a polygon are in A.P. The least angle is 120° and the common difference is 5°. Find the number of sides.

[**Hint** : Exterior angles also form an A.P. and their sum =360°.]

34. The mth term of an A.P. is n and the nth term is m. Find the rth term.

35. Find the sum of all the numbers between 100 and 700 divisible by 8.

[**Hint** : They form an A. P. with 104 as first term, 696 as last term and 8 as the common difference.]

36. Deduce an expression for the sum of natural numbers 1 to n. *(P. U. 1929)*

37. Sixty tuning forks are arranged in a row so that each gives four beats per second with the preceding one. The last fork is an octave of the first. Calculate the frequencies of the forks. *(P. U. 1936)*

CHAPTER VIII

GEOMETRICAL PROGRESSION

36. Def. Quantities are said to be in **Geometrical Progression (G P)**, when they increase or decrease by a common factor. This factor is called the **common ratio.**

Thus the following series are in G. P. :—

1. $2, 4, 8, 16, \ldots\ldots$
2. $1, \sqrt{3}, 3, 3\sqrt{3}, \ldots\ldots$
3. $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots\ldots$
4. $a, ar, ar^2, ar^3, \ldots\ldots$

The common ratios of these are $2, \sqrt{3}, \frac{1}{2}$ and r respectively. The 4th series is the general series of this type. a and r are the first term and the common ratio respectively and may have any values.

37. *To find the nth term of a G. P.*

Let $T_1, T_2, T_3, \ldots\ldots T_n$ represent the 1st, 2nd, 3rd and nth terms and let a be the first term and r the common ratio.

$$T_1 = a,$$
$$T_2 = ar,$$
$$T_3 = ar^2,$$
$$T_4 = ar^3,$$

and in general $T_n = ar^{n-1}$.

Example 1. Find the 5th term of the series $3, 2, \frac{4}{3}, \ldots\ldots$

In this series $a = 3$, $r = \frac{2}{3}$.

$$T_5 = ar^4 = 3 \times \left(\frac{2}{3}\right)^4 = \frac{16}{27}.$$

38. *To find the sum of n terms of a G. P.*

Let S_n denote the sum of n terms of a G. P. whose first term is a and common ratio is r.

$$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} \qquad \ldots(i)$$

Multiply by r

$$rS_n = ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n \qquad \ldots(ii)$$

Subtracting (*ii*) from (*i*) we get

$$S_n\ (1-r)=a \quad ar^n$$

$$\therefore \qquad S_n=\frac{a(1-r^n)}{1-r}.$$

If r is a proper fraction, *i. e.*, less than unity, each term is numerically less than the preceding one and r^n can be made sufficiently small by taking n very large. Thus, if n approaches $\propto$(infinity), r^n becomes so small that it may be neglected in comparison with 1. Thus in the above formula, if r is less than unity and n is infinitely large,

$$\text{Say } \propto=\frac{a}{1-r}. \quad (r^n \text{ has been neglected.})$$

When we say $n=\propto$ we only mean that n is approaching infinity and has got a very large value compared with other quantities.

Illustration :—

Consider the series $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots\ldots\ldots$

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{1\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}=\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}$$

When $n=100$; $r^n=\frac{1}{10^{100}}=\cdot000\ldots$ (99 zeros) 1.

when $n=1000$, $r^n=\frac{1}{10^{1000}}=\cdot000\ldots\ldots$(999 zeros) 1.

$\frac{1}{10^{100}}$ is a very small quantity indeed, but $\frac{1}{10^{1000}}$ is a great deal smaller still, yet 1000 is an ordinary number as far away from infinity as any other. If n is $=\infty$, $\frac{1}{10^n}$ will be practically zero.

Thus $S_\propto$ may be put$=\frac{1}{1-\frac{1}{10}}=\frac{10}{9}$

39. When three quantities are in G. P. the middle one is said to be the **Geometric Mean** between the other two. Thus if m is the geometric mean between a and b ; a, m b form a G.P.

$$\frac{m}{a}=\frac{b}{m}\ ;\ \therefore\ m^2=ab \text{ or } m=\pm\sqrt{ab}.$$

Geometric mean between two quantities is equal to the square root of the product of the quantities.

40. Def. When any number of quantities form a G. P., all the quantities except the first and the last are called **Geometric Means** between the first and the last quantities.

To insert n geometric means between a and b.

Let $m_1, m_2 \ldots\ldots m_n$, be the means.

$a, m_1, m_2, \ldots\ldots m_n, b$ form a G. P., with $n+2$ terms.

$b=ar^{n+1}$ (if r is the common ratio of this G. P.)

$$\therefore\quad r^{n+1}=\frac{b}{a}\ ;\ i.e.,\qquad r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$m_1=ar=a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$m_2=ar^2=a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

and so on.

Example 2. Insert 3 geometric means between 2 and 32.

If $m_1, m_2 \ldots\ldots$ are the means,

$2, m_1, m_2 \ldots\ldots 32$ form a G. P. with 5 terms.

$$32=2\times r^4,$$

$$\therefore\quad r^4=16 \text{ or } r=2.$$

$\therefore$ the Means are 4, 8 and 16.

SOLVED EXAMPLES

Example 3. In a G. P. the first term is 5, the last term is 405 and the sum is 605. Find the number of terms and the common ratio.

Let a be the first term, r the common ratio, and n the number of terms. Then the last term is the nth term and is equal to ar^{n-1}. Hence

$a=5$.........(1) ; $ar^{n-1}=405$ ... ... (1).

And since the sum is 605, therefore

$$605=a\,\frac{1-r^n}{1-r}=5.\,\frac{1-r^n}{1-r}$$

or
$$121=\frac{1-r^n}{1-r} \quad \ldots \quad \ldots \quad \ldots \ (2).$$

From (1) and (2) $r^{n-1}=81$ or $r^n=81r$. Hence substituting in (3),

$$121=\frac{1-81r}{1-r} \text{ or } 121-121r=1-81r$$

or
$$40r=120 \quad \text{or} \quad r=3.$$

Also $r^{n-1}=81$ or $3^{n-1}=81=3^4$, therefore

$n-1=4$ or $n=5$.

Hence the number of terms is 5 and the common ratio is 3.

Example 4 If the pth, qth and rth terms of a G. P. are in G. P., then p, q, r are in A. P.

Let x be the first term and y the common ratio of the given G. P. Then its

pth term $=xy^{p-1}$

qth ,, $=xy^{q-1}$

rth ,, $=xy^{r-1}$

But these three are given to be in G. P., therefore

$$\frac{xy^{q-1}}{xy^{p-1}}=\frac{xy^{r-1}}{xy^{q-1}}.$$

Simplifying,
$$y^{2q-2}=y^{p-1}y^{r-1}=y^{p+r-2}.$$

Hence
$$2q-2=p+r-2 \text{ or } q=\frac{p+r}{2}.$$

Hence q is the arithmetic mean between p and r, *i.e.*, p, q, r are in A. P.

Example 5 Show that the arithmetic mean between two quantities is greater than the geometric mean between them.

The sum of the arithmetic and geometric means between two geometric means is 2. Find the quantities.

(*P. U. 1934, 1944*)

Let a and b be the two quantities, then the Arithmetic Mean $=\frac{a+b}{2}$, and the Geometric Mean $=\sqrt{ab}$.

The arithmetic mean will be greater than the geometric mean if $\frac{a+b}{2}>\sqrt{ab}$, *i.e.*, if $a+b>2\sqrt{ab}$

i.e., if $$a+b-2\sqrt{ab}>0$$

i.e., if $$(\sqrt{a}-\sqrt{b})^2>0.$$

But the left-hand side being now a perfect square is always positive and hence >0. Therefore the arithmetic mean is greater than the geometric mean

For the second part, let a, b be the required quantities,

then $$\frac{a+b}{2}+\sqrt{ab}=18 \quad \ldots \quad \ldots(1)$$

$$\frac{a+a}{2}-\sqrt{ab}=2 \quad \ldots \quad \ldots(2)$$

From (1) and (2) by addition and subtraction, we get

$$a+b=20\ldots\ldots(3)\text{ ; and } 2\sqrt{ab}=16 \quad \ldots(4)$$

From (4), $$\sqrt{ab}=8 \text{ or } ab=64 \quad \ldots \quad \ldots(5)$$

From (3) and (5) it is now evident that

either $a=16$, $b=4$ or, $a=4$, $b=16$.

EXAMPLES VIII.

Find the

1. 6th term of 1, 3, 9,..................

2. 7th term of 1, $\frac{1}{2}$, $\frac{1}{4}$......................

3. 10th term of $\sqrt{3}$, 3, $3\sqrt{3}$,............

4. nth term of $\frac{\sqrt{2}+1}{\sqrt{2}-1}$, $\frac{1}{2-\sqrt{2}}$, $\frac{1}{2}$,............

Sum the following series :—

5. 2;—4, 8,..............................to 10 terms.

6. $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}$,..............................to 6 terms.

7. $\frac{3}{4}, 1\frac{1}{2}$, 3..............................to 5 terms.

8. 1, ·1, ·01,..............................to 10 terms.

9. $\frac{4}{3}$, 1, $\frac{3}{4}$,.. ....to infinity.

10. 3, $\sqrt{3}$, 1,..............................„ „

11. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}$..............................„ „

12. ·45, ·015, 0005,......................., „

13. Insert 5 geometric means between $\frac{32}{9}$ and $\frac{81}{2}$

14. „ 3 „ „ „ $\frac{9}{4}$ and $\frac{4}{9}$.

15. The sum of the first eight terms of a G. P. is 17 times the sum of the first four terms. Find the series.

16. In a G. P. shew that if each term be subtracted from the succeeding, the successive differences are also in G. P.

17. The 5th term of a G. P. is 81 and the 2nd term is 24. Find the series.

18. The sum of three numbers in G. P. is 21 and their product is 216. Find them.

[**Hint** : Put the numbers as $\frac{a}{r}$, a, ar and find a from the product.]

19. The sum of a G. P. whose common ratio is 3, and last term 486, is 728. Find the first term.

20. If n geometric means be inserted between a and b, show that the product of all the means$=(ab)^{\frac{n}{2}}$

21. $(m+n)$th term of a G. P. is p and $(m-n)$th term is q. Show that the mth term$=\sqrt{pq}$.

22. The sum of infinite terms of a G. P. is 4 and the sum of their cubes is 192. Find the series.

23. If the mth, nth and pth terms of a G. P. be x, y, and z respectively, prove that $x^{n-p} \times y^{p-m} \times z^{m-n}=1$.

24. The arithmetic mean of two integral numbers exceeds the geometric mean by 2 and the ratio of the two numbers is 4. Find them. (*P. U. 1928*)

25. Obtain an expression for the sum of n terms of a geometrical series. (*P. U. 1930*)

26. Show that the pressure of air inside the receiver of an air pump falls in a geometrical progression as the exhaustion proceeds. The volume of the receiver of a certain pump is 10 times the effective volume of the barrel, and 15 lb. per sq. inch is the initial pressure. Find the fall of pressure at the end of the fourth stroke. (*P. U. 1935*)

27. Deduce the formula for determining the sum of n terms in a Geometrical Progression.

Show that when a body is weighed on a false balance whose beam is horizontal without loading, the true weight is the geometric mean between its apparent weights. (*P. U. 1940*)

CHAPTER IX

MISCELLANEOUS SERIES

41. *To find the sum of the first n natural numbers.*

$$S_n = 1+2+3+\ldots\ldots n.$$

This is an A. P. with 1 as first term and 1 as common difference.

$$S_n = \frac{n}{2}\{2+(n-1)\} = \frac{n(n+1)}{2}.$$

42. *Recurring Decimals.*

To find the value of $\cdot 3\dot{2}$

$\cdot 3\dot{2} = \frac{3}{10}+\frac{2}{100}+\frac{2}{1000}$ ....to infinity

$= \frac{3}{10}+$ a geometrical series with $\frac{2}{100}$ as first term and $\frac{1}{10}$ as common ratio.

$$= \frac{3}{10}+\frac{\frac{2}{100}}{1-\frac{1}{10}} = \frac{3}{10}+\frac{2}{100}\times\frac{10}{9}$$

$$= \frac{3}{10}+\frac{2}{90} = \frac{27+2}{90} = \frac{29}{90} = \frac{32-3}{90},$$

hence the rule given in books on arithmetic.

Miscellaneous Types.

Example 1. The nth term of a series is 4^n-2n, find the sum of n terms.

Giving values 1, 2, 3,.........to n, we have

$$T_1=4^1-2\times1$$
$$T_2=4^2-2\times2$$
$$T_3=4^3-2+3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_n=4^n-2\times n$$
$$\begin{aligned}S_n&=T_1+T_2+T_3+T_4+\ldots\ldots+T_n\\&=(4+4^2+4^3+4^4+\ldots\ldots+4^n)-2(1+2+3+\ldots\ldots+n)\\&=4\frac{4^n-1}{4-1}-2\frac{n(n+1)}{2}\\&=\tfrac{4}{3}(4^n-1)-n(n+1).\end{aligned}$$

Example 2. The sum of n terms of a series is $n(29-n)$. Find the 10th term.

10th term is obviously the difference between the sum of ten terms and the sum of nine terms.

Putting $n=10$ and 9,

$$S_{10}=10(29-10)=190$$
$$S_9=(29-9)=180$$
$$\therefore\quad \text{10th term}=S_{10}-S_9=10.$$

Example 3. Sum the series $3+33+333+\ldots\ldots$to n terms.

$$\begin{aligned}S_n&=3(1+11+111+\ldots\ldots n\text{ terms})\\&=\tfrac{3}{9}(9+99+999+\ldots\ldots n\text{ terms})\\&=\tfrac{3}{9}\{(10-1)+(10^2-1)+(10^3-1)+\ldots n\text{ terms}\}.\\&=\tfrac{3}{9}\{10+10^2+10^3+\ldots\ldots n\text{ terms}-n\}.\\&=\tfrac{1}{3}\left\{\frac{10(1-10^n)}{1-10}-n\right\}=\frac{10}{27}(10^n-1)-\frac{n}{3}\end{aligned}$$

Example 4. Find the sum of n terms of the series.

$$\frac{1}{1.2}+\frac{1}{2.3}+\ldots\ldots+\frac{1}{n(n+1)}.$$

Here
$$T_1=\frac{1}{1.2}=1-\tfrac{1}{2}.$$
$$T_2=\frac{1}{2.3}=\tfrac{1}{2}-\tfrac{1}{3},$$
$$T_3=\frac{1}{3.4}=\tfrac{1}{3}-\tfrac{1}{4},$$

....

$$T_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Adding all these equations,

$$S_n=T_1+T_2+\ldots+T_n=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

Example 5. Find the nth term of the series

$$3+6+18+66+\ldots\ldots\ldots$$

Let T_n denote the nth term of the series and S_n its sum to n terms. Then

$$S_n=3+6+18+66+\ldots\ldots\ldots+T_n$$

Again $$S_n=\quad 3+6+18+\ldots\ldots\ldots+T_{n-1}+T_n.$$

Subtracting,

$$0=3+3+12+48+\ldots\ldots\ldots+(T_n-T_{n-1})-T_n$$

or $$T_n=3+\{3+12+48+\ldots\ldots\ldots+(T_n-T_{n-1})\},$$
$$=3+\{3+12+48+\ldots\ldots\text{to }(n-1)\text{ terms}\}$$
$$=3+3\{1+4+16+\ldots\ldots\text{to }(n-1)\text{ terms}\}$$
$$=3+3\frac{4^{n-1}-1}{4-1}.$$
$$=3+4^{n-1}-1$$
$$=4^{n-1}+2.$$

43. Prove that if a body starting with a velocity u and having an acceleration a, travels a distance s in time t,

$$s=ut+\tfrac{1}{2}at^2.$$

Acceleration a means that after each second the velocity increases by a.

If the velocity at start is u, the velocity after one second will be $u+a$, after 2 seconds it will be $u+2a$ and in general after t seconds it will be $u+at$. If final velocity is denoted by v,

$$v=u+at.$$

Let the time t be divided into a very large number of intervals, say n. Each interval $=\frac{t}{n}$ sec.

Velocity at the beginning of the first interval $=u$

Velocity at the end of the first interval $=u+a\frac{t}{n}$ [$\because v=u+at$]

Velocity at the beginning of the second interval $=u+a\times\frac{t}{n}$

Velocity at the end of the second interval $=u+a\times\frac{2t}{n}$

Velocity at the beginning of the third interval $=u+a\times\frac{2t}{n}$

Velocity at the end of the third interval $=u+a\times\frac{3t}{n}$

As the velocity is gradually changing at a uniform rate, average velocity during an interval obviously lies between the velocities at the beginning and the end of the interval. Distance travelled is equal to the product of the average velocity and time, therefore the distance travelled in the first interval lies between $u\times\frac{t}{n}$ and $\left(u+a\frac{t}{n}\right)\frac{t}{n}$, i.e. between $\frac{ut}{n}$ and $\frac{ut}{n}+a\frac{t^2}{n^2}$.

Similarly the distance travelled in the second interval lies between

$$\frac{ut}{n}+a\frac{t^2}{n^2} \text{ and } \frac{ut}{n}+2a\frac{t^2}{n^2},$$

the distance travelled in the third interval lies between

$$\frac{ut}{n}+2a\frac{t^2}{n^2} \text{ and } \frac{ut}{n}+3a\frac{t^2}{n^2}.$$

Total distance travelled in n intervals thus lies between

$$\left(\frac{ut}{n}\right)+\left(\frac{ut}{n}+\frac{at^2}{n^2}\right)+\left(\frac{ut}{n}+2a\frac{t^2}{n^2}\right)+\ldots\ldots n \text{ terms}$$

and $$\left(\frac{ut}{n}+\frac{at^2}{n^2}\right)+\left(\frac{ut}{n}+2a\frac{t^2}{n^2}\right)+\left(\frac{ut}{n}+3a\frac{t^2}{n^2}\right)+\ldots n \text{ terms.}$$

viz., between $n\times\frac{ut}{n}+a\frac{t^2}{n^2}\left(1+2+3+\ldots+\overline{n-1}\right)$

and $n\times\frac{ut}{n}+\frac{at^2}{n^2}(1+2+3+\ldots+n)$

i.e., between $ut+\frac{at^2}{n^2}\times\frac{(n-1)\,(n)}{2}$ and $ut+\frac{at^2}{n^2}\times\frac{n(n+1)}{2}$

i.e., between $ut+\frac{at^2\left(1-\frac{1}{n}\right)\times 1}{1}$ and $ut+\frac{at^2\times 1\times\left(1+\frac{1}{n}\right)}{2}$

By making n very large we can make $\frac{1}{n}$ as small as we please, and thus it can be neglected. Both the limits become $ut+\frac{1}{2}at^2$.

$\therefore$ the total distance travelled lies between $ut+\frac{1}{2}at^2$ and $ut+\frac{1}{2}at^2$ *viz.*, it is $=ut+\frac{1}{2}at^2$.

EXAMPLES IX

1. Find the sum of first n (*i*) odd, (*ii*) even natural numbers.

2. Add $x+a+x^2+2a+x^3+3a+\ldots n$ terms where n is even.

[**Hint** : There are two different series, therefore the sum is equal to the sums of $\frac{n}{2}$ terms of each of these two series. Questions 3, 4, and 5 are somewhat similar.]

3. $a(a+b)+a^2(a^2+b^2)+a^3(a^3+b^3)+\ldots n$ terms.

4. $\frac{4}{9}-\frac{5}{9^2}+\frac{4}{9^3}-\frac{5}{9^4}+\frac{4}{9^5}-\frac{5}{9^6}+\ldots\ldots$to infinity.

5. Find the sum of 20 terms of the series.

$$4+7+9+12+14+17+\ldots\ldots$$

6. Find the nth term and the sum of n terms of the series

$$a+2(a+d)+3(a+2d)+4(a+3d)+\dots\dots$$

7. nth term of a series is 3^n-2^n. Find the sum of n terms.

8. Find the sum of n terms of $4+44+444+\dots\dots$

9. The sum of n terms of a series is $\frac{1}{2}(11n^2+9n)$. Find the 6th term.

10. The sum of n terms of a series is $a+bn+cn^2$. Find the pth term.

11. The sum of n terms of a series is $2(2^n-1)$. Find the 8th term.

12. Find the value of $\dfrac{1+2+3+\dots\dots n}{n^2}$ where $n=\infty$.

13. Show that $3^{\frac{1}{2}}\times 9^{\frac{1}{4}}\times 27^{\frac{1}{8}}\times\dots\dots$ad infinitum$=9$.

[**Hint** : Logarithms of the factors of the product form an arithmetico-geometric series.]

14. Find the sum of 15 terms of the series

4, 6, 12, 30, 84,...... *(P.U. 1931)*

15. Find the sum of 40 terms of the series :—

$1+3+4+6+7+9+10+\dots\dots$ *(P.U. 1931)*

16. A body starting with velocity u and acceleration a travels for n small intervals of time, each equal to T. Supposing that during each interval the velocity remains equal to its value at the beginning of that interval, find the total distance travelled.

Also show that the value of this distance travelled reaches the limit $ut+\frac{1}{2}at^2$ when n becomes very large, where $nT=t$.

(P.U. 1932)

CHAPTER X

BINOMIAL THEOREM

44. The formula

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\times 2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\times 2\times 3}a^3x^{n-3}+\dots$$

is known as the *Binomial Theorem.*

It is also called the expansion of $(x+a)^n$. This theorem was first discovered by Newton and it is true for all positive integral values of n.

45. *To prove the Binomial Theorem.*

We know that

$$(x+a)^2=x^2+2ax+a^2$$

$$(x+a)^3=x^3+3ax^2+3a^2x+a^3.$$

These can be written in the forms

$$(x+a)^2=x^2+\frac{2}{1}ax+\frac{2\times1}{1\times2}a^2.$$

$$(x+a)^3=x^3+\frac{3}{1}ax^2+\frac{3\times2}{1\times2}a^2x+\frac{3\times2\times1}{1\times2\times3}a^3.$$

These forms agree with the Binomial theorem given above, therefore the theorem is true when $n=2$ or $n=3$.

Suppose it is true for some general value of n, *viz.,*

$$(x+a)^n=x^n+\frac{n}{1}ax^{n-1}+\frac{n(n-1)}{1\times2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\times2\times3}a^3x^{n-3}+ \qquad \ldots(i)$$

Multiplying both sides by $(x+a)$, we get,

$$(x+a)(x+a)^n=x^{n+1}+\frac{n}{1}ax^n+\frac{n(n-1)}{1\times2}a^2x^{n-1}$$

$$+\frac{n(n-1)(n-2)}{1\times2\times3}a^3x^{n-2}+\ldots\ldots+ax^n+\frac{n}{1}a^2x^{n-1}$$

$$+\frac{n(n-1)}{1\times2}a^3x^{n-2}+\ldots\ldots\ldots$$

Adding similar terms together, we get

$$(x+a)^{n+1}=x^{n+1}+\frac{n+1}{1}ax^n+\frac{(n+1)n}{1.2}a^2x^{n-1}$$

$$+\frac{(n+1)n(n-1)}{1.2.3}a^3x^{n-2}+\ldots\ldots\ldots\ldots(ii)$$

Comparing (i) and (ii) we find that (ii) can be obtained from (i) by substituting $n+1$ for n.

This shows that if the Binomial Theorem is true for the index n, it is also true for the index $n+1$.

We have seen by trial that the theorem is true for index 2 and also for index 3, therefore it is also true for index 4. If it is true for index 4, it is also true for index 5 and so on. Thus it is true generally for any positive integral index.

46. Put $x=1$ and $a=x$ in the formula of Art. 44.

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2\,3}x^3+\ldots\ldots$$

Writing $-x$ for x, we have

$$(1-x)^n=1-nx+\frac{n(n-1)}{1.2}x^2-\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots$$

These forms are most frequently employed and should be carefully remembered.

The proof of the Binomial Theorem for negative and fractional indices is beyond the scope of this book, but it is true in these forms when $x<1$.

The co-efficient of 2nd term in these expansion is $\frac{n}{1}$ and the index of x is 1.

Co-eff. of 3rd term is $\frac{n(n-1)}{1.2}$ and the index of x is 2.

Co-eff. of 4th term is $\frac{n(n-1)(n-2)}{1.2.3}$ and the index of x is 3.

In general the co-efficient of $(r+1)$th term is

$$\frac{n(n-1)\ldots\{n-(\overline{r+1}-2)\}}{1.2.3\ldots\ldots r}$$

and the index of x is r,

$\therefore$ T_{r+1} of these expansions is $\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}x^r$

(r+1)th term is called the general term.

Rule. *In the numerator there are r factors beginning with n and decreasing by unity. In the denominator also there are r factors, the first r natural numbers. The index of x is also r.*

NOTE. The sign (positive or negative of x) must be kept with x.

47. Consider the expansion

$$(1+x)^6=1+6x+\frac{6\cdot 5}{1\cdot 2}x^2+\frac{6.5.4}{1.2.3}x^3+\frac{6.5.4.3}{1.2.3.4}x^4+\frac{6.5.4.3.2}{1.2.3.4.5}x^5+\frac{6.5.4.3.2.1}{1.2.3.4.5.6}x^6$$

$$=1+6x+15x^2+20x^3+15x^4+6x^5+x^6.$$

Note that

(*i*) the total number of terms is one more than the index and

(*ii*) the co-efficients of terms equidistant from the beginning and end are equal.

It is more convenient in almost all cases to express the quantity to be expanded in the form $(1+x)^n$. The calculation of co-efficients after the middle term in case of positive integral values of n is thus saved.

Example 1. Expand $(2a-\frac{2}{3})^8$.

$$(2a-\tfrac{2}{3})^8=\left\{2a\left(1-\frac{1}{3a}\right)\right\}^8=2^8a^8\left(1-\frac{1}{3a}\right)^8$$

$$=256a^8\left\{1-\frac{8}{3a}+\frac{8.7}{1.2}\left(-\frac{1}{3a}\right)^2+\frac{8.7.6}{1.2.3}\left(-\frac{1}{3a}\right)^3+\frac{8.7.6.5}{1.2.3.4}\left(-\frac{1}{3a}\right)^4\right.$$

$$\left.+\frac{{}^*8.7.6}{1.2.3}\left(-\frac{1}{3a}\right)^5+\frac{8.7}{12}\left(-\frac{1}{3a}\right)^6+8\left(-\frac{1}{3a}\right)^7+\left(-\frac{1}{3a}\right)^8\right\}$$

$$=256a^8\left\{1-\frac{8}{3a}+\frac{28}{9a^2}-\frac{56}{27a^3}+\frac{70}{81a^4}-\frac{56}{243a^5}+\frac{28}{729a^6}\right.$$

$$\left.-\frac{8}{2187a^7}+\frac{1}{6561a}\right\}$$

$$=256\left\{a^8-\frac{8a^7}{3}+\frac{28a^6}{9}-\frac{56a^5}{27}+\frac{70a^4}{81}-\frac{56a^3}{243}+\frac{28a^2}{729}-\frac{8a}{2187}+\frac{1}{6561}\Big)\right\}$$

*There are nine terms in this expansion. Fifth is the middle term and after this the co-efficients have been written in the reverse order.

SOLVED EXAMPLES

Example 2. Expand $(3x+2y)^5$.

$$(3x+2y)^5=(3x)^5+5(2y)(3x)^4+\frac{5.4}{1.2}(2y)^2(3x)^3+\frac{5.4.3}{1.2.3}(2y)^3(3y)^2$$

$$+\frac{5.4.3.2}{1.2.3.4}(2y)^4(3x)+\frac{5.4.3.2.1}{1.2.3.4.5}(2y)^5$$

$$=243x^5+810x^4y+1080x^3y^2+720x^2y^3+240xy^4+32y^5.$$

Example 3. Write down the 5th term of $(a^2-b^2)^{12}$.

$$(a^2-b^2)^{12}=a^{24}\left(1-\frac{b^2}{a^2}\right)^{12}$$

In the fifth term of $\left(1-\frac{b^2}{a^2}\right)^{12}$ there will be 4 factors in the numerator and 4 in the denominator, and the index of $\left(-\frac{b^2}{a^2}\right)$ will be 4.

$$\text{Hence the 5th term}=a^{24}\left\{\frac{12\times11\times10\times9}{1\times2\times3\times4}\left(-\frac{b^2}{a^2}\right)^4\right\}$$

$$=a^{24}\times495\,\frac{b^8}{a^8}=495a^{16}b^8.$$

Example 4. Find the expansions of $\frac{1}{1-x}$ and $\frac{1}{1+x}$.

$$\frac{1}{1-x}=(1-x)^{-1}$$

$$=1+(-1)(-x)+\frac{(-1)(-2)}{1.2}(-x)^2+\frac{(-1)(-2)(-3)}{1.2.3}(-x)^3$$

$$+\frac{(-1)(-2)(-3)(-4)}{1.2.3.4}(-x)^4+\ldots\ldots$$

$$=1+x+x^2+x^3+x^4+\ldots\ldots\ldots\ldots(i)$$

Similarly $\frac{1}{1+x}=(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots\ldots(ii)$

Forms (*i*) and (*ii*) are important.

When x is small that x^2, x^3, etc., are negligible, the expansions become $1+x$ and $1-x$ respectively.

Example 5. Expand $(1+x)^{\frac{2}{3}}$ to four terms ; $x<1$.

$$(1+x)^{\frac{2}{3}}=1+\tfrac{2}{3}x+\frac{\frac{2}{3}(\frac{2}{3}-1)}{1.2}x^2+\frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)x^3}{1.2.3}+\ldots\ldots\ldots\ldots$$

$$=1+\tfrac{2}{3}x-\tfrac{1}{9}x^2+\tfrac{4}{81}x^3.$$

Example 6. Expand $(4a-8x)^{-\frac{1}{2}}$ to four terms; given $a>2x$.

$$(4a-8x)^{-\frac{1}{2}}=(4a)^{-\frac{1}{2}}\left[1-\frac{2x}{a}\right]^{-\frac{1}{2}}$$

$$=\frac{1}{2\sqrt{a}}\left[1-\frac{2x}{a}\right]^{-\frac{1}{2}}$$

$$=\frac{1}{2\sqrt{a}}\left\{1+(-\tfrac{1}{2})\frac{(-2x)}{a}+\frac{-\frac{1}{2}(-\frac{1}{2}-1)}{1.2}\left[\frac{-2x}{a}\right]^2\right.$$

$$\left.+\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}\left[\frac{-2x}{a}\right]^3\right\}$$

$$=\frac{1}{2\sqrt{a}}\left\{1+\frac{x}{a}+\frac{3x^2}{2a^2}+\frac{5x^3}{2a^3}\right\}.$$

Example 7. Find the third term in the expansion of $(1-x)^{-4}$.

$$T_3=\frac{(-4)(-4-1)}{1.2}x^2=10x^2.$$

48. An interesting application of the binomial theorem is given below.

Example 8. Find the value of $\sqrt[3]{1001}$ correct to four decimal places.

$$(1001)^{\frac{1}{3}}=(1000+1)^{\frac{1}{3}}=[1000(1+\tfrac{1}{1000})]^{\frac{1}{3}}=10(1+{\cdot}001)^{\frac{1}{3}}$$

$$=10\left\{1+\tfrac{1}{3}\times{\cdot}001+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1.2}({\cdot}001)^2+\ldots\ldots\right\}$$

$$=10[1+{\cdot}000333-{\cdot}0000001]$$

$$=10+{\cdot}00333-{\cdot}000001$$

$=10{\cdot}0033$ upto four decimal places.

Example 9. Evaluate $\sqrt[4]{626}$ correct to four places of decimals.

$$(626)^{\frac{1}{4}}=(625+1)^{\frac{1}{4}}=(5^4+1)^{\frac{1}{4}}=5\left(1+\frac{1}{5^4}\right)^{\frac{1}{4}}$$

$$=5\left(1+\frac{1}{625}\right)^{\frac{1}{4}}=5(1+\cdot 0016^{*})^{\frac{1}{4}}$$

$$=5\left\{1+\tfrac{1}{4}\times\cdot 0016+\frac{\frac{1}{4}(\frac{1}{4}-1)}{1.2}\times\cdot 00000256+\ldots\ldots\ldots\right.$$

$$=5\{1+\cdot 0004-\cdot 00000024\}$$

$$=[5+\cdot 0020+\cdot 00000120]$$

$=5+\cdot 0020$ to four decimal places.

*The value of $\frac{1}{5^4}$ has been found in decimals easily as under :

$$\frac{1}{5^4}=\frac{1}{5^4}\times\frac{2^2}{2^2}=\frac{2^4}{10^4}=\frac{16}{10000}=\cdot 0016.$$

EXAMPLES X

Expand the following :

1. $(1-3a^2)^6$. **2.** $(2x-y)^5$. **3.** $(5-4x)^4$.

4. $\left(1-\frac{1}{a}\right)^{10}$. **5.** $\left(y+\frac{1}{y}\right)^5$.

Expand to four terms and also find their approximate values when z is so small that its square and higher powers may be neglected.

6. $\sqrt{1+x}$. **7.** $\left(1+\frac{2x}{3}\right)^3$. **8.** $\left(1+\frac{x}{3}\right)^{-3}$.

9. $(1-3x)^{-\frac{1}{3}}$ **10.** $(1-2x)^{-\frac{3}{2}}$. **11.** $\frac{1}{(1-x)^2}$.

12. Find the 4th term in the expansion of $(x-5)^{13}$.

13. ,, ,, 5th ,, ,, ,, ,, ,, $\left(2a-\frac{b}{3}\right)^3$.

14. Find the 11th term in the expansion of $(1-2x^8)^{\frac{11}{2}}$.

15. ,, ,, 5th ,, ,, ,, ,, ,, $(3a-2b)^{-1}$

16. Find the middle term in the expansion of $\left(1-\frac{x^2}{2}\right)^{14}$.

17. Find the value of $\sqrt[5]{33}$ correct to four places of decimals. Check your result by logarithms.

18. Obtain the square root of 999998 without actually extracting it. (*P. U. 1929*)

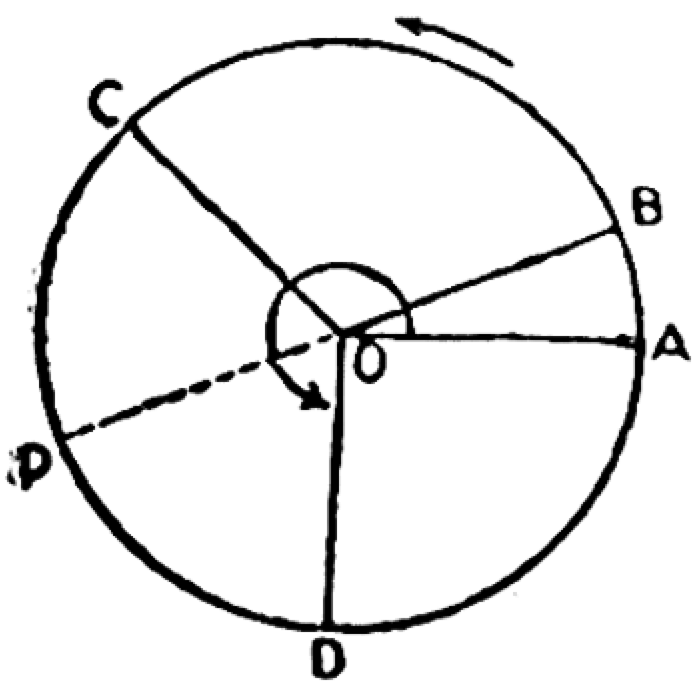
19. State the Binomial Theorem for all real values of n, and apply it to find the value of $(1{\cdot}002)^5$ correct to four decimal places. (*P. U. 1941*)

20. State the Binomial Theorem for a Positive Integral Index. Calculate 19^4 from the relation $19^4 = (20-1)^4$ (*P. U. 1942*)

CHAPTER XI

TRIGONOMETRY

SEXAGESIMAL AND CIRCULAR MEASURES OF ANGLES



49. Definition of an angle. Suppose a line OP revolves about the point O in a *counter clockwise* direction and passes from OA, its position of start, successively through positions OB, OC, OD, etc., then the angle between OA and any position like OD is measured by the **amount of revolution** which the line has undergone from OA, the position of start, to OD, its final position. Moreover, the line may have taken up position OD after making any number of complete revolutions through the position OA, therefore the angle may be of **any magnitude**.

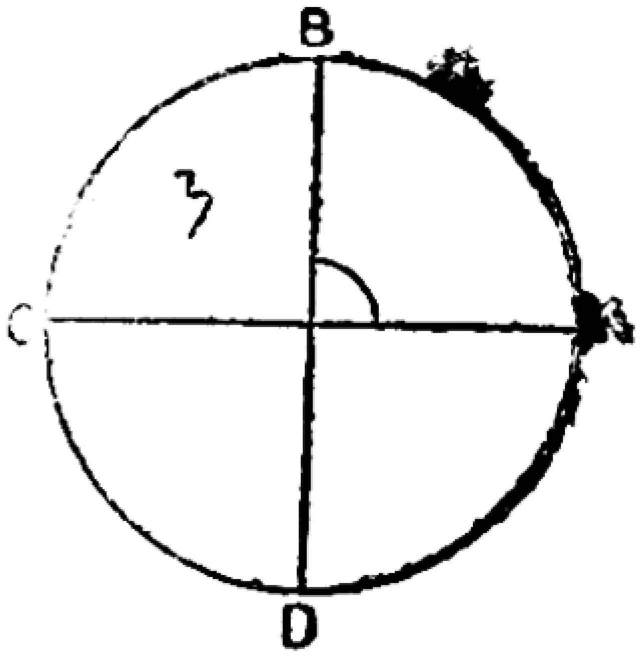
50. For measurement of a quantity some unit is always selected, and essentials of a unit are :—

(*i*) It must have a constant value in order that the measurements of one observer may agree with those of any other, anywhere.

(*ii*) It must be easily procurable.

E.g., a yard is the unit of length and it has been defined by an Act of Parliament as the distance between two marks at a certain temperature (62° F) on a Bronze bar preserved safely in the Office of the Board of Trade. A large number of copies of this length have been made and are available everywhere. If yards were of different lengths one can imagine the confusion arising out of it.

Angles are measured in three systems of units, but only two of them will be described here.



51. Sexagesimal Measure. The angle described by the revolving line in one complete revolution is divided into four equal part, each part being called a **right angle.** A right angle is further divided into 90 equal parts called **degrees**. A degree is the unit in this system.

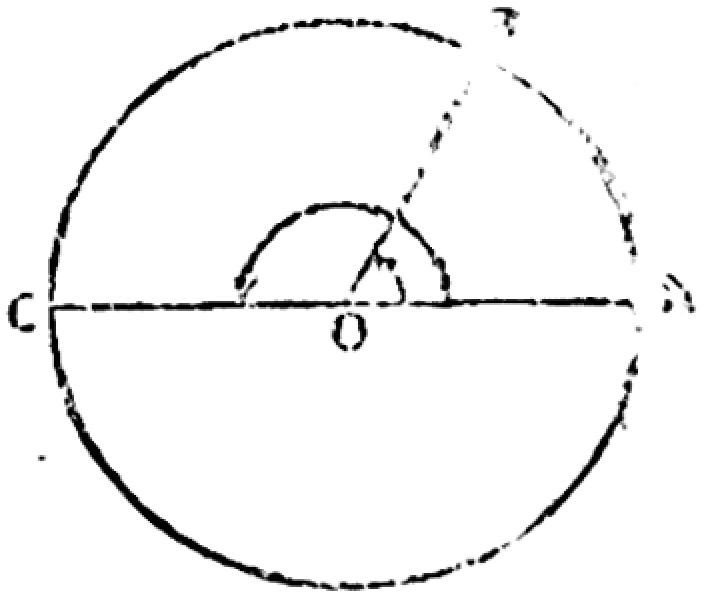
A circle is easily drawn anywhere by a pair of compasses and the circumference is divided into four equal parts at points A, B, C and D. These points joined to the centre give four angles, each being a right angle. The amount of rotation in one complete revolution is the same, whatever the radius of the circle, therefore a right angle which is equal to $\frac{1}{4}$th of the complete revolution has a constant value independent of the length of the radius of the circle described. A degree is $\frac{1}{90}$th of a right angle, therefore this has also a constant value. Thus the requisites of a unit are satisfied. A degree is sub-divided into 60 equal parts called **minutes** and a minute again into 60 equal parts called **seconds**. For brevity degrees, minutes and seconds are denoted by the symbols (°) degrees, (′) minutes and (″) seconds. Thus $10^\circ\, 2'\, 37''$ means 10 degrees, 3 minutes and 37 seconds.

Protractors having degrees marked on them are available and any angle can be measured by them.

52. Circular measure The unit in this system is a radian. A **radian** is defined as the angle subtended at the centre of a circle by an *arc* equal in length to the radius of the **circle.**

53. *To prove that a radian has a constant value.*

• Draw a circle with O as centre and any radius (*r*). Let an arc AB be equal in length to OA, then angle AOB is a radian.



(**Caution.** The length of the *arc* AB, not the chord, is equal to OA.)

Produce AO to meet the circumference at C. From geometry we know that the angles subtended at the centre of a circle are proportional to the arcs subtending them, therefore

$$\frac{\angle AOB}{2 \text{ rt. angles}}=\frac{\text{Arc AB}}{\text{semi-circumference}}=\frac{r}{\pi r}=\frac{1}{\pi},$$

$$\therefore \ \angle AOB=\frac{2 \text{ rt. angles}}{\pi}$$

But $\angle AOB$ is a radian

$$\therefore \ \text{A radian}=\frac{2\text{rt. angles}}{\pi}=\frac{180^\circ}{\pi}.$$

The numerator and denominator of the right hand side are both constant, therefore a radian has a constant value independent of the size of the circle.

[We have assumed that the circumference of a circle $=2\pi r$ where $\pi=3{\cdot}142$ correct to three figures. This relation is true and has been verified in more than one way.]

54. The relation, $\frac{180}{\pi}$ =**one radian**, proved in the last article gives the method of converting degrees into **radians** and *vice versa.*

$$\text{A radian } \frac{180^\circ}{\pi}=\frac{180}{3{\cdot}142}=57^\circ{\cdot}29 \text{ approx},$$

π *radians* $=180^\circ$ is the general form of **expressing this relation.**

The student should commit to memory the values of the following angles in radian measure.

$360^\circ=2\pi$ **radians.**

$180^\circ=\pi$ „ $\qquad$ $90^\circ=\frac{\pi}{2}$**radians.**

$60^\circ = \frac{\pi}{3}$ radians $\qquad 45^\circ = \frac{\pi}{4}$ radians

$30^\circ = \frac{\pi}{6}$,,

Example 1. Convert 42° 7′ 45″ into radians.

$$42^\circ\ 7'\ 45'' = 42^\circ\ 7\tfrac{3}{4}' = 42^\circ\frac{7\cdot75}{60} = 42^\circ\cdot129$$

$$180^\circ : 42^\circ\cdot129 :: \pi : x$$

$$\therefore\ x = \frac{42\cdot129 \times 3\cdot142}{180} = \cdot7354 \text{ radians}$$

$$\begin{array}{r} 1\cdot6246 \\ \cdot4972 \\ \hline 2\cdot2553 \\ \hline 1\cdot8665 = \cdot7354. \end{array}$$

Logarithmic process is shown on the right.

Example 2. Express 1·309 radians in sexagesimal measure.

$$\pi : 1\cdot309 :: 180^\circ : x$$

$$\therefore\quad x = \frac{1\cdot309 \times 180}{3\cdot142} = 75^\circ.$$

55. The sexagesimal system does not possess the advantages of a decimal system, therefore in many physical measurements the angles are measured in decimals of a degree. The change from one form into the other is simple.

Example 3. Express 3° 37′ 12″ in decimal fractions.

Divide 12″ by 60 to convert into minutes.

Add 37′ and divide again by 60 to make degrees.

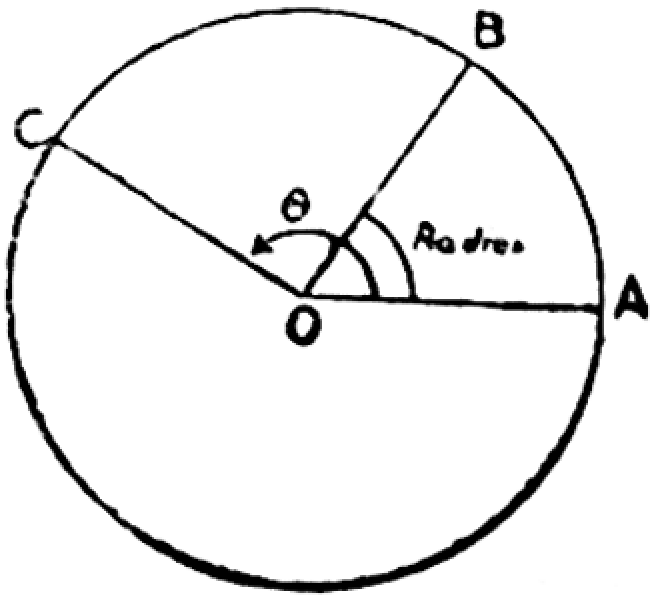
Add the degrees.

For the opposite change the process is reversed as illustrated in the margin.

$$\begin{array}{ll} 60 & \underline{/12 \text{ seconds.}} \\ & \quad \cdot2 \\ \hline & \quad 37 \\ 60 & \underline{|37\cdot2} \\ & \quad \cdot62 \\ \hline & \quad 3 \\ & \quad 3\cdot62 \text{ degrees} \\ \hline & \quad 3^\circ\cdot62 \\ & \quad \underline{60} \\ & \quad 37'\cdot2 \\ & \quad \underline{60} \\ & \quad 12''\cdot0 \\ & = 3^\circ 37'\ 12'' \end{array}$$

56. *To prove that* $\theta = \dfrac{l}{r}$.

where θ is the circular measure (value in radians) of an angle subtended at the centre, by an arc l, of a circle of radius r.



Let AOC be the angle. Cut an arc AB equal to OA.

∠AOB = a radian.

If θ is the radian measure of angle AOC,

$$\theta = \frac{\angle AOC}{\angle AOB} = \frac{\text{arc } AC}{\text{arc } AB} = \frac{l}{r}$$

$$\theta = \frac{l}{r} \text{ or } l = r\theta.$$

If any two of these three quantities are given the third can be found.

This important relation affords a practical method of measuring an angle in circular measure. A circle of any radius enclosing the given angle is drawn. The length of the arc is measured by thread or dividers and the ratio of the arc to the radius gives the required value.

Example 4. Find the angle subtended at the centre of a circle of radius 10 cm. by an arc 23 cm. in length.

$$\theta = \frac{l}{r} = \frac{23}{10} = 2{\cdot}3 \text{ radians or } 2{\cdot}3 \times \frac{180}{3{\cdot}124} = 132^\circ{\cdot}1.$$

Example 5. A horse is tethered to a stake by a rope 27 ft. long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far it will have gone when the rope has traced out an angle of 70°. ($\pi = \frac{22}{7}$).

(P. U. 1941)

Here the radius = 27 ft. and the angle subtended by the arc at the centre of the circle

$$70^\circ = \frac{70\pi}{180} \text{ radians} = \frac{70}{180} \times \frac{22}{7} \text{ radians} = \frac{11}{9} \text{ radians.}$$

Hence the length of arc = radius × angle subtended at the centre

$$= 27 \times \frac{11}{9} = 33 \text{ ft},$$

Hence the horse moves through 33 ft.

EXAMPLES XI

(Use logarithms for calculations, $\pi=\cdot142$.)

Express in circular measure.

1. 37° 51′. **2.** 112° 43.′ **3.** 25° 50′. **4.** 15°.

Express the following radians in sexagesimal measure.

5. $\frac{2}{3}$. **6.** ·7. **7.** 1·508. **8.** $\frac{3\pi}{8}$. **9.** $\frac{7\pi}{12}$.

10. If the radius of a circle be 25 ft. find the length of an arc which subtends an angle equal to 3″ at the centre.

11. Find by using the formula $\theta=\frac{l}{r}$, the number of revolutions made in one minute by the wheel of a bicycle of adius $1\frac{1}{4}$ ft. travelling at the rate of 10 miles an hour.

12. Find the number of radians in the angles of a triangle, which are in A. P. and the greatest of which is 105°.

13. Find the length of a line on the surface of the earth, subtending an angle of 2′ 30″ at its centre [radius of the earth =4000 miles].

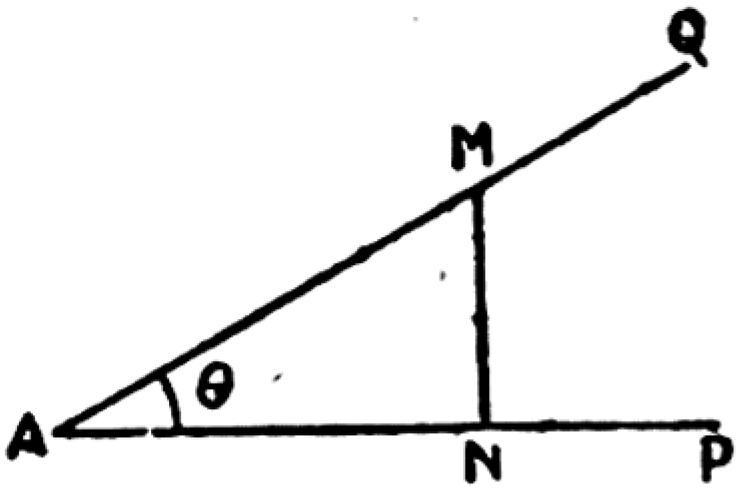
14. A train is running round a circular curve of 5 miles radius and a man stationed at the centre finds that the minute hand of his watch once pointed to the train exactly keeps pace with it. Find the speed of the train.

15. Find the angle traversed in one minute by the spoke of a carriage wheel of radius 2 ft. travelling at a speed of 10 miles an hour.

CHAPTER XII

Trigonometrical Ratios

57. Trigonometrical ratios of an acute angle.



Let PAQ (θ) be an acute angle. Take any point **M** on AQ draw MN perpendicular to AP A right-angled triangle MAN is formed, which AM is the hypotenuse. The side AN adjacent to the angle (θ), under consideration, is called the *base*, and MN, the side opposite to the angle (θ) is called *height*.

The ratio $\frac{MN}{AM}$ or $\frac{\text{height}}{\text{hypotenuse}}$ is known as **sine** of θ or simply as **sin** θ.

The ratio $\frac{AN}{AM}$ or $\frac{\text{base}}{\text{hypotenuse}}$ is known as **cosine** of θ o simply as **cos** θ.

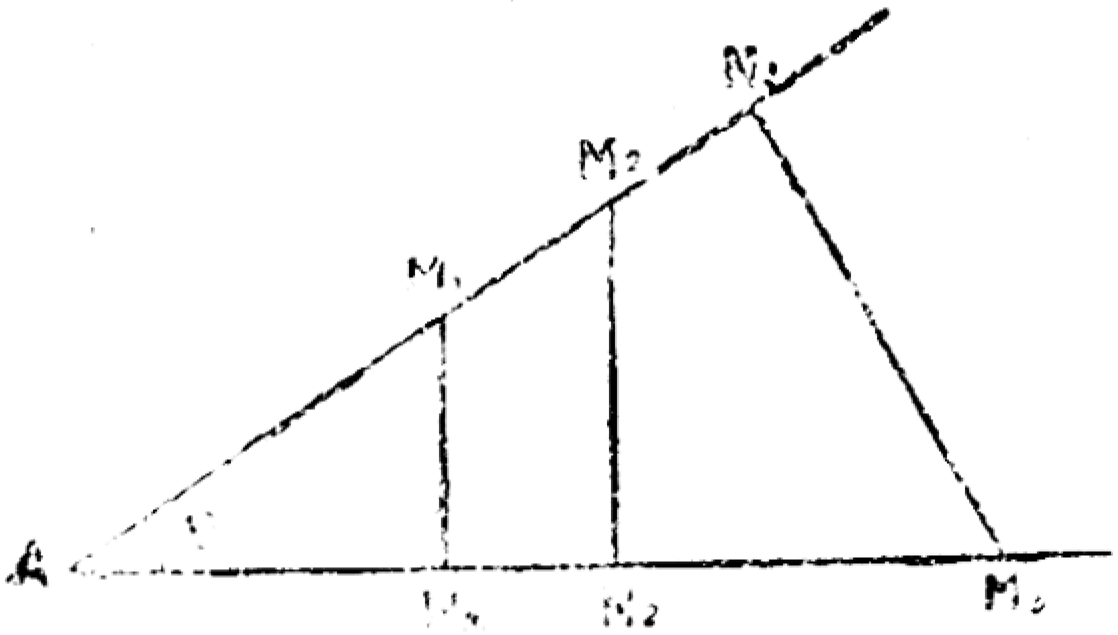
The ratio $\frac{MN}{AN}$ or $\frac{\text{height}}{\text{base}}$ is known as **tangent** of θ or simply as **tan** θ.

The ratio $\frac{AN}{MN}$ or $\frac{\text{base}}{\text{height}}$ is known as **cotangent** of or simply as **cot** θ.

The ratio $\frac{AM}{AN}$ or $\frac{\text{hypotenuse}}{\text{base}}$ is known as **secant** of θ or simply as **sec** θ.

The ratio $\frac{AM}{MN}$ or $\frac{\text{hypotenuse}}{\text{height}}$ is known as **cosecant of** θ or simply as **cosec** θ.

These six ratios are called **trigonometric or circular functions of** θ.



58. It can be shown that whatever the position of M may be, these ratios are constant for a given angle. For, let three different points M_1, M_2, and M_3 be taken on one or the other bounding line of an angle θ. Let M_1N_1, M_2N_2, M_3N_3 be the perpendiculars to the other line.

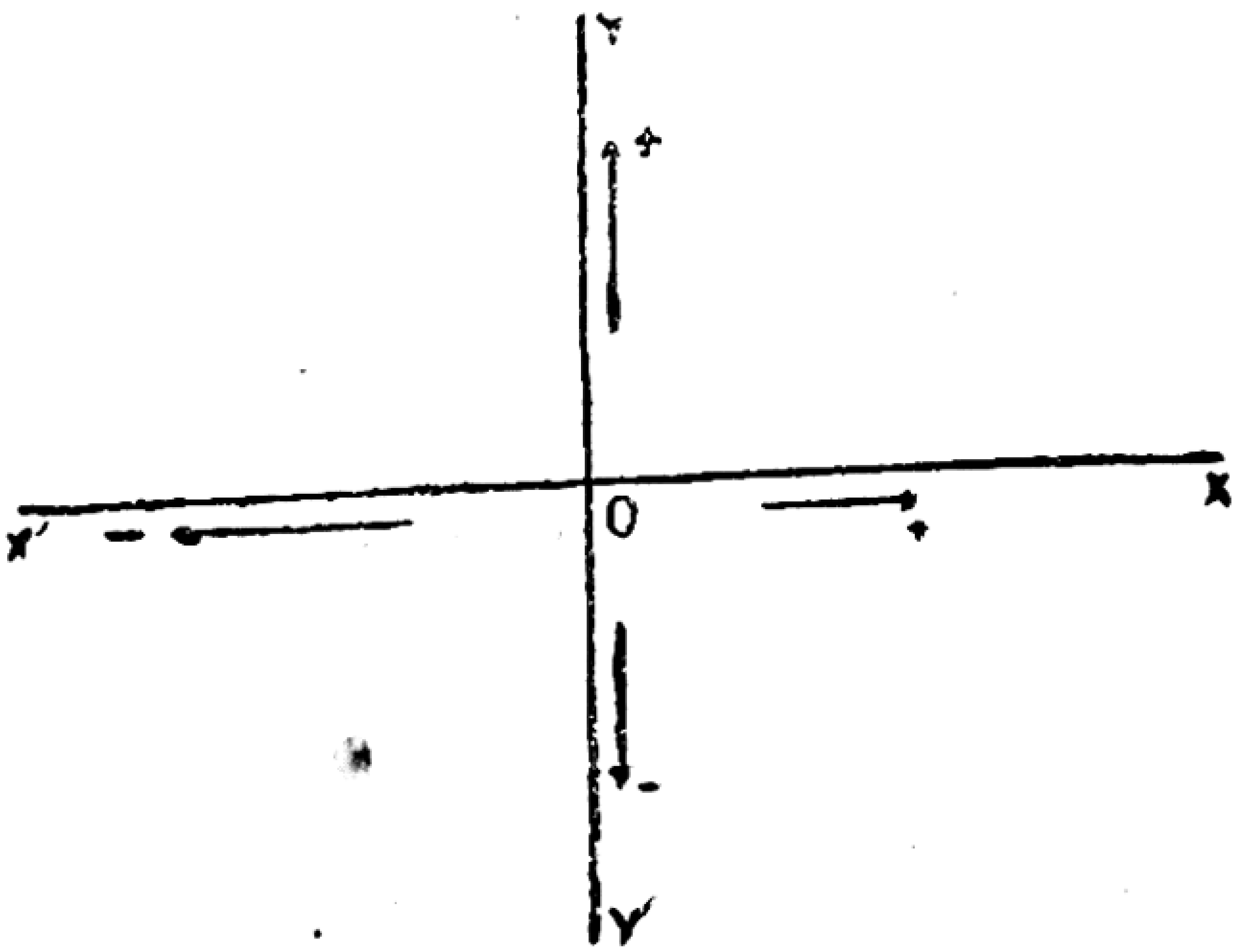
△s M_1AN_1, M_2AN_2 and M_3AN_3 are similar because they are all right-angled triangles having one acute angle, common, therefore

$$\frac{M_1N_1}{AM_1}=\frac{M_2N_2}{AM_2}=\frac{M_3N_3}{AM_3}$$

All these ratios represent sin θ as defined above, therefore the value of sin θ is independent of the position of M. It depends upon the magnitude of the angle only.

Similarly the values of the other circular functions cos, tan, cot, etc., of the angle can also be proved to be constant.

59. Conventions of signs and quadrants. Take two lines XOX′ and YOY′ perpendicular to each other intersecting at O.



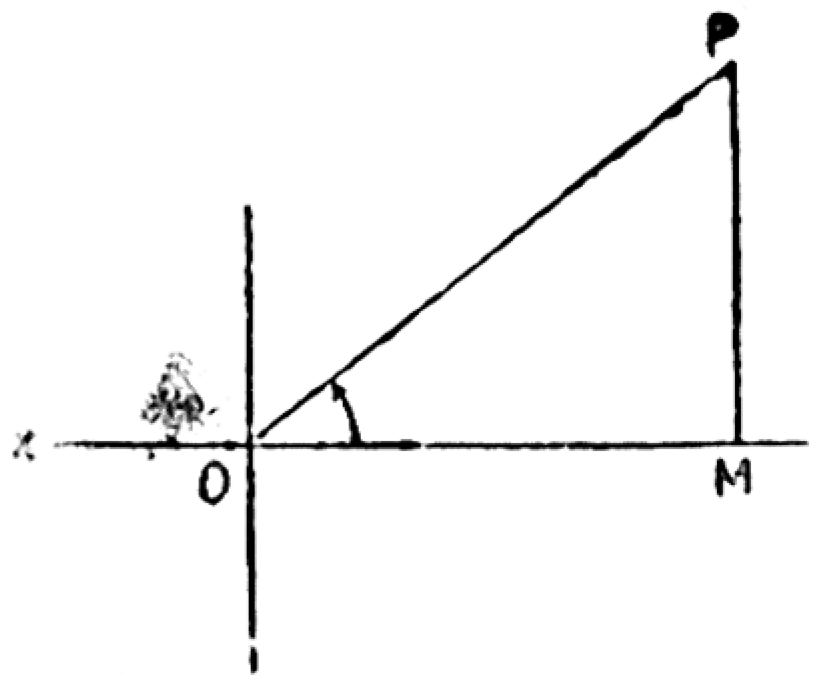
Distances in the direction OX (to the right) are taken as positive.

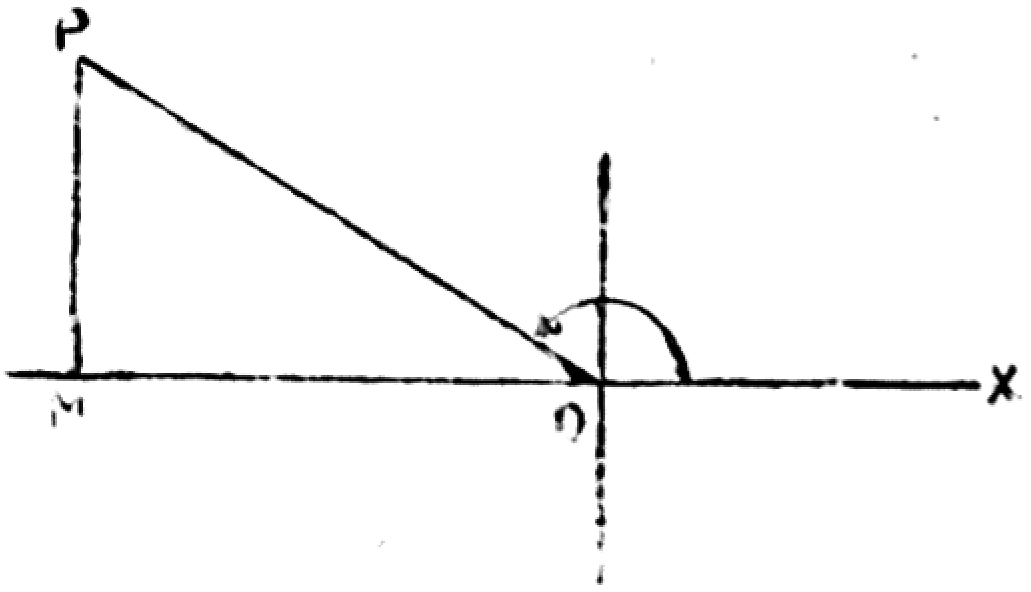
Distances in the direction OX′ (to the left) are taken as negative.
Distances in the direction OY (upwards) are taken as positive.
Distances in the direction OY′ (downwards) are taken as negative.
Space between the lines OX and OY is called the *first quadrant.*
Space between the lines OY and OX′ is called the *second* ,, .
Space between the lines OX′ and OY′ is called the *third* ,, .
Space between the lines OY′ and OX is called the *fourth* ,, .
Lines XOX′ is called the axis of X and YOY′ the axis of Y.

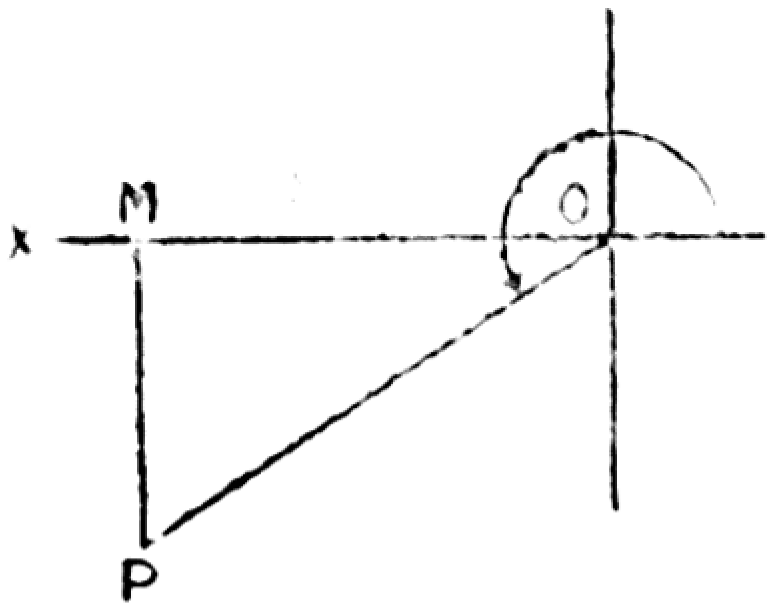
If we take a point P (see fig. of the next article) anywhere and draw PM perpendicular to OX ; OM is called the *abscissa* and MP. the *ordinate* of P.

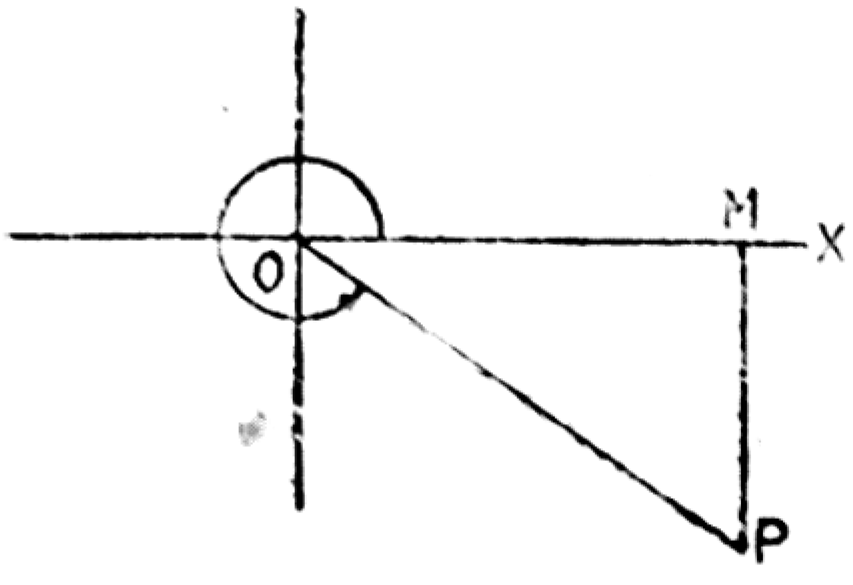
60. Trigonometrical ratios of an angle of any magnitude.

Let θ be an angle of any magnitude. The revolving line OP known as radius vector starts from OX and after making any









number of complete revolutions takes up a position OP in any one of the four quadrants.

Draw PM the ordinate of P.

$$\sin\theta \text{ is defined as } \frac{\text{ordinate}}{\text{radius vector}}=\frac{MP}{OP}.$$

$$\cos\theta \quad ,, \quad ,, \quad \frac{\text{abscissa}}{\text{radius vector}}=\frac{OM}{OP}.$$

$$\tan\theta \quad ,, \quad ,, \quad \frac{\text{ordinate}}{\text{abscissa}}=\frac{MP}{OM}.$$

$$\cot\theta \quad ,, \quad ,, \quad \frac{\text{abscissa}}{\text{ordinate}}=\frac{OM}{MP}.$$

$$\sec\theta \quad ,, \quad ,, \quad \frac{\text{radius vector}}{\text{abscissa}}=\frac{OP}{OM}.$$

$$\operatorname{cosec}\theta \quad ,, \quad ,, \quad \frac{\text{radius vector}}{\text{ordinate}}=\frac{OP}{MP}.$$

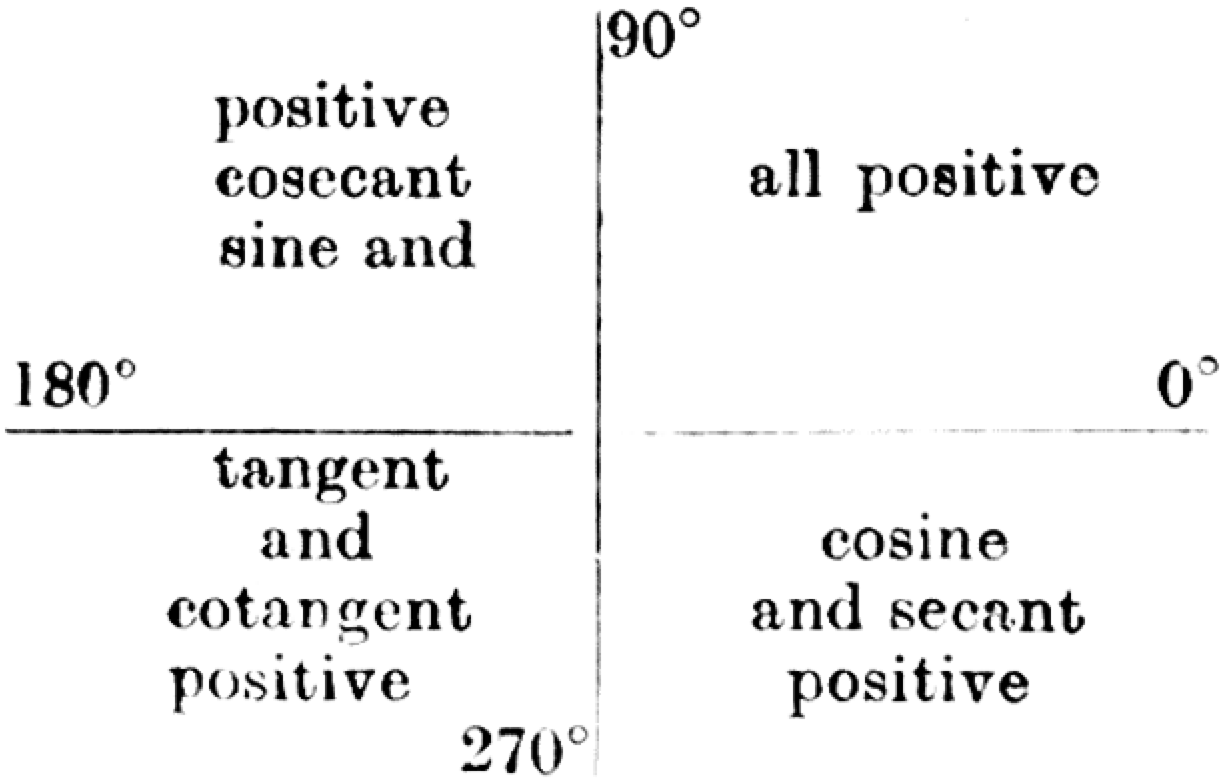
61. (*i*) In the first quadrant, the ordinate, the abscissa and the radius vector are all positive, therefore **all these ratios** are positive.

[**Note**. Radius vector is always taken as positive, in whichever quadrant it may lie.]

(*ii*) In the second quadrant the abscissa OM is negative and the others *viz.*, MP and OP are positive, therefore **sine** and **cosecant**, which are the ratios between two positive quantities are **positive**, while cosine, tangent, cotangent and secant being the ratios of one positive and one negative quantity are negative.

(*iii*) In the third quadrant the radius vector is positive while the abscissa and the ordinate are both negative. **Tangent** and **cotangent** have both numerator and denominator negative, therefore they **are positive**. Sine, cosine, secant, and cosecant have one of these positive and the other negative, therefore they are negative.

(*iv*) In the fourth quadrant the abscissa and the radius vector are positive and the ordinate is negative, therefore **cosine and secant are positive** and the rest are negative.



These results are summarized in the above figure, which will help greatly in remembering them.

62. In order to find out in which quadrant a given angle lies we should remember, that for angles less than four right angles the limits of the 1st, 2nd, 3rd and 4th quadrants are 0°—90° ; 90°—180° ; 1·0°—270° ;and 270° to 360° respectively. If the angle is greater than 360°, substract the greatest multiple of 360° from it then the remaining angle being less than 360°can be easily located in one of the four quadrants.

Example 1. Will cos 280° be positive or negative ?

280° is >270° and <360°, therefore it lies in the fourth quadrant. Thus cos 280° is positive.

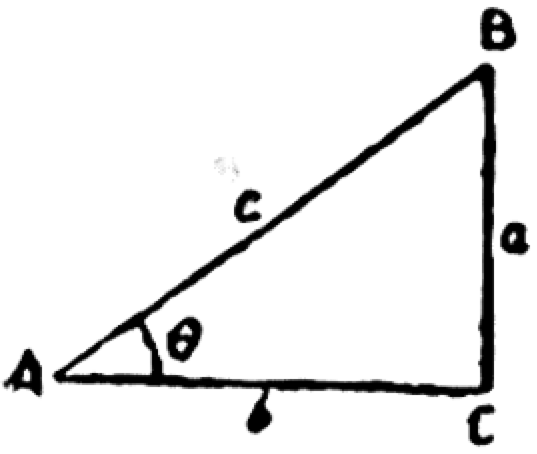
Example 2. Which of the circular functions of 750° are positive ?

750°=2×360°+30°, ∴ it lies in the first quadrant.

All the circular functions of 750° are hence positive.

SIMPLE RELATIONS BETWEEN TRIGONOMETRICAL RATIOS

63. A glance at the definitions of these ratios shows that



(i) $\text{cosec}\,\theta=\dfrac{1}{\sin\theta}$

$\left[\because \text{cosec}\,\theta=\dfrac{c}{a} \text{ and } \sin\theta=\dfrac{a}{c}\right]$

(ii) $\sec\theta=\dfrac{1}{\cos\theta}$ $\left[\because \sec\theta=\dfrac{c}{a} \text{ and } \sin\theta=\dfrac{b}{c}\right]$

(iii) $\cot\theta=\dfrac{1}{\tan\theta}$ $\left[\because \cot\theta=\dfrac{b}{a} \text{ and } \tan\theta=\dfrac{a}{b}\right]$

The sine, cosine and tangent are called the **principal ratios**, because the other three can be easily derived from them, being merely the reciprocals of these three.

64. *To prove that* $\frac{\sin\theta}{\cos\theta}=\tan\theta$ and $\frac{\cos\theta}{\sin\theta}=\cot\theta$.

We have $$\frac{\sin\theta}{\cos\theta}=\frac{\frac{a}{c}}{\frac{b}{c}}=\frac{a}{c}\times\frac{c}{b}=\frac{a}{b}=\tan\theta\ ;$$

$$\frac{\cos\theta}{\sin\theta}=\frac{\frac{b}{c}}{\frac{a}{c}}=\frac{b}{c}\times\frac{c}{a}=\frac{b}{a}=\cot\theta.$$

65. *To prove that* :—

(*i*) $\sin^2\theta+\cos^2\theta=1$.

(*ii*) $\sec^2\theta=1+\tan^2\theta$.

(*iii*) $\operatorname{cosec}^2\theta=1+\cot^2\theta$

[**Note**. $(\sin\theta)^2$, $(\tan\theta)^2$ etc., are written as $\sin^2\theta$, $\tan^2\theta$, etc.]

We have $a^2+b^2=c^2$ [ABC being a right-angled triangle.

(*i*) $$\sin^2\theta+\cos^2\theta=\left(\frac{a}{c}\right)^2+\left(\frac{b^2}{c}\right.=\frac{a^2+b^2}{c^2}=1.$$

(*ii*) $$\sec^2\theta=\left(\frac{c}{b}\right)^2=\frac{c^2}{b^2}=\frac{a^2+b^2}{b^2}=1+\frac{a^2}{b^2}$$

$$=1+\left(\frac{a}{b}\right)^2=1+\tan^2\theta.$$

(*iii*) $$\operatorname{cosec}^2\theta=\left(\frac{c}{a}\right)^2=\frac{c^2}{a^2}=\frac{a^2+b^2}{a^2}=1+\frac{b^2}{a^2}$$

$$=1+\left(\frac{b}{a}\right)^2=1+\cot^2\theta.$$

66. The following forms of $\sin^2\theta+\cos^2\theta=1$ are important.

$\sin^2\theta=1-\cos^2\theta$ or $\sin\theta=\sqrt{1-\cos^2\theta}$

$\cos^2\theta=1-\sin^2\theta$ or $\cos\theta=\sqrt{1-\sin^2\theta}$.

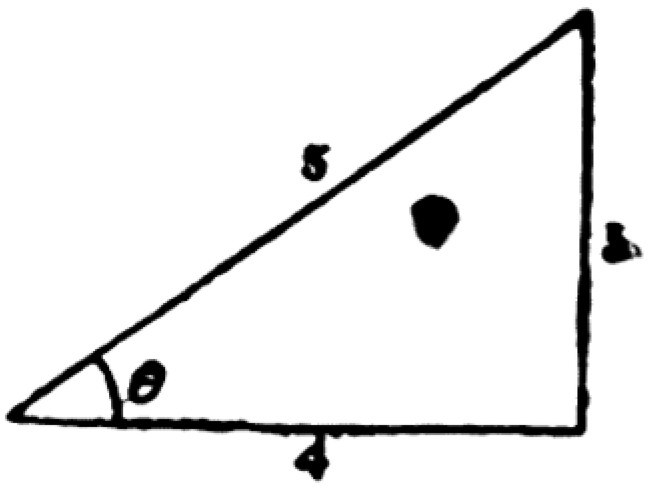
Example 3. Prove that $\cot^2\theta\ (1-\cos^2\theta)=\cos^2\theta$.

$$\cot^2\theta\ (1-\cos^2\theta)=\frac{\cos^2\theta}{\sin^2\theta}\ (\sin^2\theta)=\cos^2\theta.$$

67. All the trigonometrical ratios of an angle can be found if any one of them is given.

Example 4. Find all the trigonometrical ratios of an acute angle whose tangent $=\frac{3}{4}$.

Draw a right-angled triangle having its height equal to 3 and base equal to 4.



Let θ be the given angle, then

$$\tan\ \theta=\frac{\text{height}}{\text{base}}=\frac{3}{4}$$

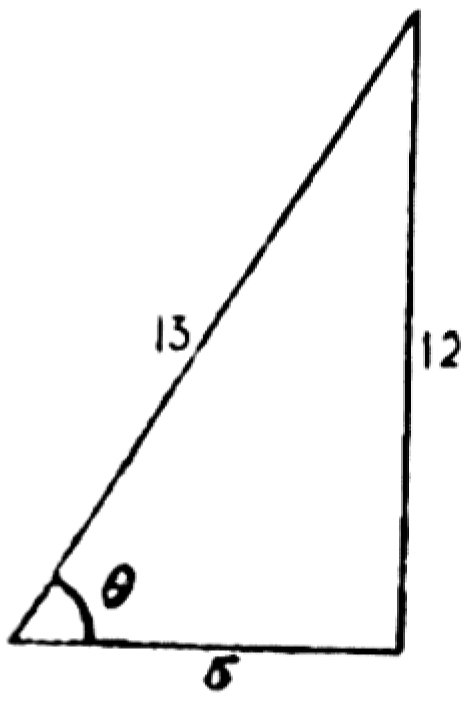
Hypotenuse $=\sqrt{3^2+4^2}=5$.

(The angle is acute, $\therefore$ only positive value of the square root has been taken.)

All the trigonometrical ratios of θ can be written down from the figure.

$\sin\ \theta=\frac{3}{5}$, $\cos\ \theta=\frac{4}{5}$, $\cot\ \theta=\frac{4}{3}$, $\sec\ \theta=\frac{5}{4}$, $\operatorname{cosec}\ \theta=\frac{5}{3}$.

There are other methods also of solving such questions, but the one given above, *viz.*, of drawing a right-angled triangle and giving suitable values of the corresponding sides is the simplest.



Example 5. $\cos\ \theta=\frac{5}{13}$; find $\tan\ \theta$.

Let 5 and 13 be the values of the base and the hypotenuse of a right-angled triangle.

$$\text{Height}=\sqrt{13^2-5^2}=12,$$

$$\therefore\quad \tan\ \theta=\frac{12}{5}.$$

68 Inverse Functions.

The relation $\sin\ \theta=x$ can also be written as $=\sin^{-1}x$. $\sin^{-1}x$ is read as 'inverse sine of x' and it stands for an angle (θ) whose sine is equal to x.

Similarly $\cos^{-1}x$, $\tan^{-1}x$, mean angles whose cosine and tangent respectively are equal to x.

Example 6. Find geometrically the value of $\cos^{-1}\frac{3}{4}$.

Here we have to find an angle whose cosine is equal to $\frac{3}{4}$. This can be done by a construction similar to Art. 67, drawing a circle of radius four units, cutting a length OA on the axis of X equal to 3 units and drawing perpendicular to it.

SOLVED EXAMPLES

Example 7. Prove that

$$\tan\theta\,(1-\cot^2\theta)+\cot\theta\,(1-\tan^2\theta)=0.$$

$$\tan\theta\,(1-\cot^2\theta)+\cot\theta\,(1-\tan^2\theta)$$

$$=\tan\theta\left(1-\frac{1}{\tan^2\theta}\right)+\frac{1}{\tan\theta}\,(1-\tan^2\theta)$$

$$=\tan\theta-\frac{1}{\tan\theta}+\frac{1}{\tan\theta}-\tan\theta=0.$$

Example 8. Prove that

$$(\sin A-\cos A)\,(\sec A+\operatorname{cosec} A)=\tan A-\cot A.$$

$$(\sin A-\cos A)\,(\sec A+\operatorname{cosec} A)$$

$$=(\sin A-\cos A)\left(\frac{1}{\cos A}+\frac{1}{\sin A}\right)$$

$$=(\sin A-\cos A)\left\{\frac{\sin A+\cos A}{\sin A\cos A}\right\}$$

$$=\frac{\sin^2 A-\cos^2 A}{\sin A\cos A}=\frac{\sin^2 A}{\sin A\cos A}-\frac{\cos^2 A}{\sin A\cos A}$$

$$=\frac{\sin A}{\cos A}-\frac{\cos A}{\sin A}=\tan A-\cot A.$$

EXAMPLES XII

1. Which of the trigonometric functions of the following angles are positive?

(*i*) 290°. (*ii*) 190°. (*iii*) 135°. (*iv*) 620°.

(*v*) 780°. (*vi*) $\frac{4\pi}{3}$. (*vii*) $\frac{9\pi}{4}$. (*viii*) $\frac{7\pi}{4}$.

Prove that

2. $\cot A \sec A \sin A = 1$. 3. $\cos\theta \tan\theta = \sin\theta$.

4. $(1-\sin^2\theta)\sec^2\theta = 1$. 5. $(1-\cos^2\theta)\sec^2\theta = \tan^2\theta$.

6. $\operatorname{cosec}^2 A + \sec^2 A = \operatorname{cosec}^2 A \sec^2 A$.

7. $\tan\alpha + \cot\alpha = \sec\alpha \operatorname{cosec}\alpha$.

8. $\cos\beta \sqrt{\cot^2\beta+1} = \sqrt{\operatorname{cosec}^2\beta - 1}$.

$\sin A \sin B \cot B = \cos A \cos B \tan A$.

10. $\dfrac{1}{\sec^2 A} + \dfrac{1}{\operatorname{cosec}^2 A} = 1$.

11. $\sec^4 A - 1 = 2\tan^2 A + \tan^4 A$.

12. $\cos^2 A(\sec^2 A - \tan^2 A) + \sin^2 A(\operatorname{cosec}^2 A - \cot^2 A) = 1$.

13. $(\cos\alpha - \sin\alpha)(\operatorname{cosec}\alpha - \sec\alpha) = \operatorname{cosec}\alpha \sec\alpha - 2$.

14. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$.

15. $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

16. $\cos^2 A - \sin^2 A = \cos^4 A - \sin^4 A$.

17. $\dfrac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} + \dfrac{\cos\alpha - \cos\beta}{\sin\alpha + \sin\beta} = 0$.

18. $(\tan A - 1)^2 + (\cot A - 1)^2 = (\sec A - \operatorname{cosec} A)^2$,

19. If $\sin\theta = \frac{7}{25}$, find $\tan\theta$ and $\sec\theta$.

20. If $2\tan\theta = 1$, find $\sec\theta$ and $\operatorname{cosec}\theta$.

21. If the secant of an angle is 7, find its sine and cotangent.

22. Given : $\tan\theta = \dfrac{2mn}{m^2-n^2}$, find $\cos\theta$ and $\operatorname{cosec}\theta$.

23. If $\sec\alpha = \frac{13}{5}$, find the value of $\dfrac{2\sin\alpha - 3\cos\alpha}{4\sin\alpha - 9\cos\alpha}$.

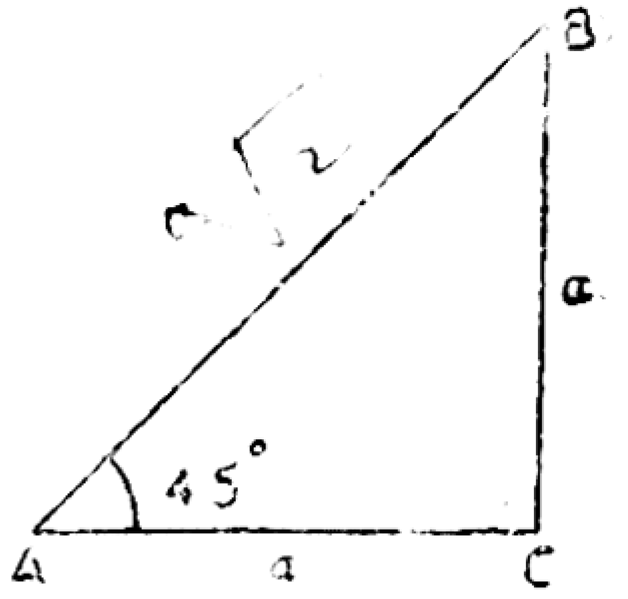
CHAPTER XIII

CIRCULAR FUNCTIONS OF 45°, 60°, 30°, 0° and 90°

69. Circular Functions of 45° or $\frac{\pi}{4}$.

Let BAC be an isosceles right-angled triangle. Each of the angles A and B=45° and side AC=BC.

If AC is$=a$, AB$=\sqrt{a^2+a^2}=a\sqrt{2}$.



$$\sin 45^\circ=\frac{a}{a\sqrt{2}}=\frac{1}{\sqrt{2}},\quad \operatorname{cosec} 45^\circ=\sqrt{2}.$$

$$\cos 45^\circ=\frac{a}{a\sqrt{2}}=\frac{1}{\sqrt{2}},\quad \sec 45^\circ=\sqrt{2}.$$

$$\tan 45^\circ=\frac{a}{a}=1,\quad \cot 45^\circ=1.$$

The three principal ratios only need be remembered.

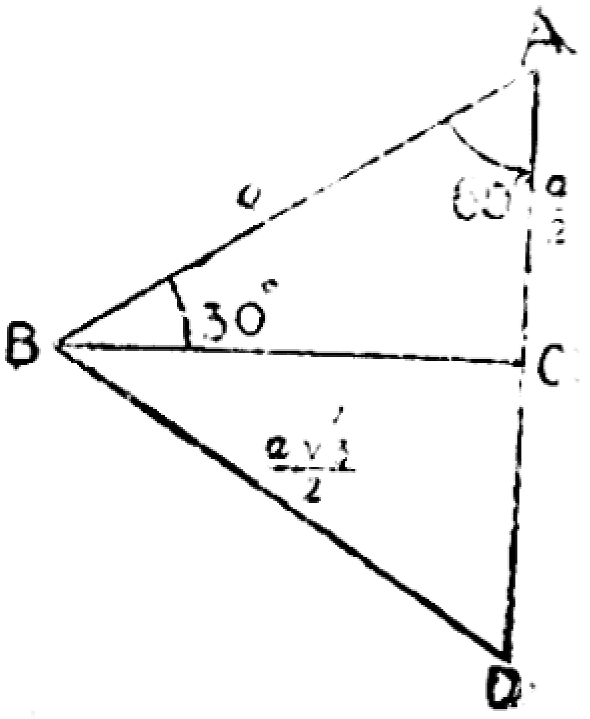
70. Circular Functions of 30° or $\frac{\pi}{6}$.

Let ABD be an equilateral triangle with sides$=a$.

Draw BC perpendicular to AD.

C is the middle point of the side AD and BC bisects the ∠ABD.

$$AC=\frac{a}{2} \text{ and } \angle ABC=30^\circ.$$



$$BC=\sqrt{AB^2-AC^2}=\sqrt{a^2-\frac{a^2}{4}}=\sqrt{\frac{3a^2}{4}}=\frac{a\sqrt{3}}{2}.$$

Consider the right-angled-triangle ABC,

$$\sin 30^\circ=\frac{\frac{a}{2}}{a}=\tfrac{1}{2}.\quad \operatorname{cosec} 30^\circ=2.$$

$$\cos 30^\circ=\frac{\frac{a\sqrt{3}}{2}}{a}=\frac{\sqrt{3}}{2},\quad \sec 30^\circ=\frac{2}{\sqrt{3}}.$$

$$\tan 30^\circ = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}. \qquad \cot 30^\circ = \sqrt{3}.$$

71. Circular functions of 60° or $\frac{\pi}{3}$.

In the figure of the last article, ABC is a right-angled triangle with angle $A = 60^\circ$.

$$\sin 60^\circ = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}, \quad \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}},$$

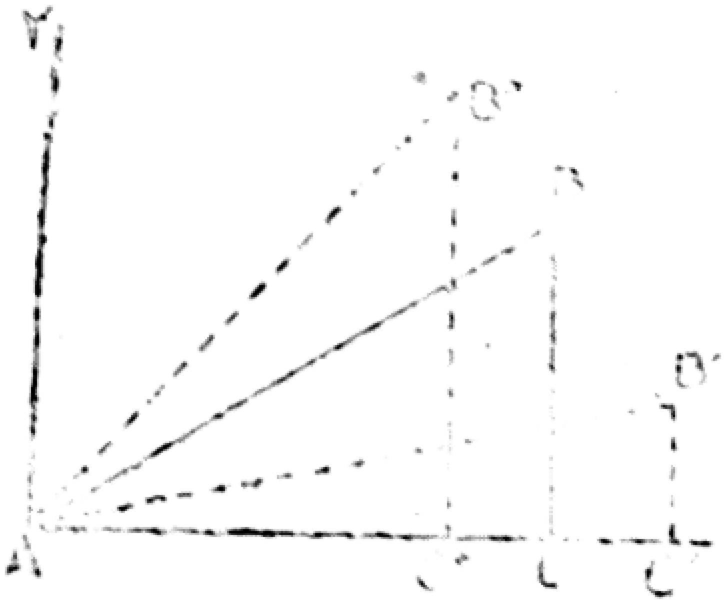
$$\cos 60^\circ = \frac{\frac{a}{2}}{a} = \tfrac{1}{2}, \qquad \sec 60^\circ = 2$$

$$\tan 60^\circ = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3}. \quad \cot 60^\circ = \frac{1}{\sqrt{3}}.$$

72. Circular functions of 0°.

Let BAC be a right-angled triangle.

As angle A diminishes, BC diminishes and AC increases un when $\angle A = 0^\circ$, AB coincides with AC and B coincides with C, $BC = 0$ and $AC = AB$.



$$\sin 0^\circ = \frac{BC}{AB} = \frac{0}{AB} = 0, \quad \operatorname{cosec} 0^\circ = \infty.$$

$$\cos 0^\circ = \frac{AC}{AB} = \frac{AB}{AB} = 1 \quad \sec 0^\circ = 1.$$

$$\tan 0^\circ = \frac{BC}{AC} = \frac{0}{AB} = 0, \quad \cot 0^\circ = \infty.$$

73. Circular functions of 90° or $\frac{\pi}{2}$.

In the figure of the last article as $\angle A$ increases, BC increases and AC diminishes, until when $A = 90^\circ$, $BC = AB$ and $AC = 0$

$$\sin 90^\circ=\frac{BC}{AB}=\frac{AB}{AB}=1, \qquad \operatorname{cosec} 90^\circ=1.$$

$$\cos 90^\circ=\frac{AC}{AB}=\frac{0}{AB}=0, \qquad \sec 90^\circ=\infty.$$

$$\tan 90^\circ=\frac{BC}{AC}=\frac{AB}{0}=\infty, \qquad \cot 90^\circ=0.$$

74. The functions of 0°, 30°, 45°, 60° and 90° occur frequently and it is important that the student should be able to write down the values of at least the principal functions of these angles at once from memory.

Example 1. Find the numerical value of

$$\cot 60^\circ \tan 30^\circ+\sec^2 45^\circ.$$

The given expression $=\dfrac{1}{\tan 60^\circ}\times\tan 30^\circ+\dfrac{1}{\cos^2 45^\circ}$

$$=\frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}}+(\sqrt{2})^2=\tfrac{1}{3}+2=2\tfrac{1}{3}.$$

Example 2. Solve $\quad 4\sin\alpha+4\cos^2\alpha=5.$

$$4\sin\alpha+4\cos^2\alpha=5$$

$$\therefore\quad 4\sin\alpha+4(1-\sin^2\alpha)=5$$

or $$4\sin^2\alpha-4\sin\alpha+1=0.$$

$$\therefore\quad \sin\alpha=\frac{4\pm\sqrt{16-16}}{8}=\tfrac{1}{2}. \qquad \therefore\quad \alpha=30^\circ.$$

[NOTE. α may also have a value in the second quadrant but unless otherwise stated angles in the first quadrant only will be considered in this book.]

Example 3. Restricting yourself to acute angles only find in degrees the value of—

$$\sin^{-1}\tfrac{1}{2}+\cos^{-1}\frac{\sqrt{3}}{2}-\tan^{-1}1+\sin^{-1}\frac{\sqrt{3}}{2}.$$

$\sin^{-1}\frac{1}{2}$ means an angle whose sine $=\frac{1}{2}$, *i.e.*, it is equal to 30°.

Similarly $\cos^{-1}\dfrac{\sqrt{3}}{2}=30^\circ$; $\tan^{-1}1=45^\circ$; and $\sin^{-1}\dfrac{\sqrt{3}}{2}=60^\circ$.

$\therefore$ the given expression $=30^\circ+30^\circ-45^\circ+60^\circ=75^\circ$.

EXAMPLES XIII.

Evaluate

1. $(\sin 60^\circ - \sin 45^\circ)(\cos 30^\circ + \cos 45^\circ)$.

2. $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$.

3. $\cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} + \frac{3}{4}\tan^2\frac{\pi}{6} + \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{6}$.

4. $\frac{1}{2}\operatorname{cosec}^2 60^\circ + \sec^2 45^\circ - 2\cot^2 60^\circ$.

5. $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$.

6. $\tan^2 60^\circ + 4\cos^2 45^\circ + 3\sec^2 30^\circ$.

7. If $\tan^2 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ$, find x.

Find the value of the acute angle α in the following equations :

8. $\tan^2\alpha + \frac{2}{\sqrt{3}}\tan\alpha + 1 = 2$.

9. $\tan\alpha + \cot\alpha = 2$.

10. $8\sin^4\alpha - 6\sin^2\alpha + 1 = 0$.

11. $2\cos^2\alpha + 11\sin\alpha = 7$.

Taking acute angles only, find the value in degrees of

12. $\cos^{-1} 0 + \sin^{-1}\frac{1}{\sqrt{2}} - \cos^{-1}\frac{1}{2}$.

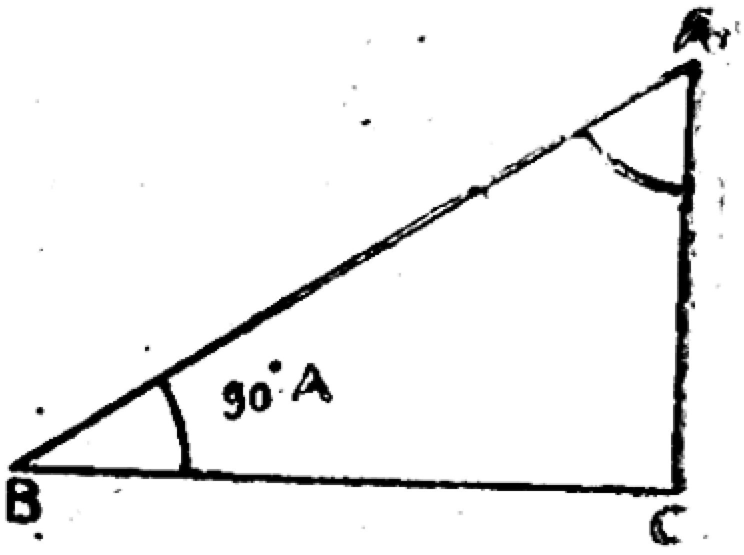
13. $\tan^{-1}\frac{1}{\sqrt{3}} + \cos^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0 - \tan^{-1}\sqrt{3}$.

14. $\sin^{-1}\frac{1}{2} + \tan^{-1} 0 + \sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{\sqrt{2}}$.

CHAPTER XIV

Functions of Allied Angles

75. Def. When the sum of any two angles is equal to a right angle, each is said to be the **complement** of the other and the angles are said to be **complementary**. Thus angles 90°$-$A and A are complementary and angle 90°$-$A is the complement of angle A.



76. Functions of complementary angles, *viz.*, **A** and **(90°$-$A).**

Take ABC a right-angled triangle.

∠s A and R are complementary, $\therefore$ **90°$-$A=.B.**

$$\textbf{sin } (90°-A)=\sin B=\frac{AC}{AB}=\textbf{cos A}, \qquad \text{[Def.]}$$

$$\textbf{cos } (90°-A)=\cos B=\frac{BC}{AB}=\textbf{sin A.}$$

$$\textbf{tan } (90°-A)=\tan B=\frac{AC}{BC}=\textbf{cot A.}$$

Similarly it can be proved that

$$\cot (90°-A)=\tan A.$$
$$\sec (90°-A)=\operatorname{cosec} A.$$
$$\operatorname{cosec} (90°-A)=\sec A.$$

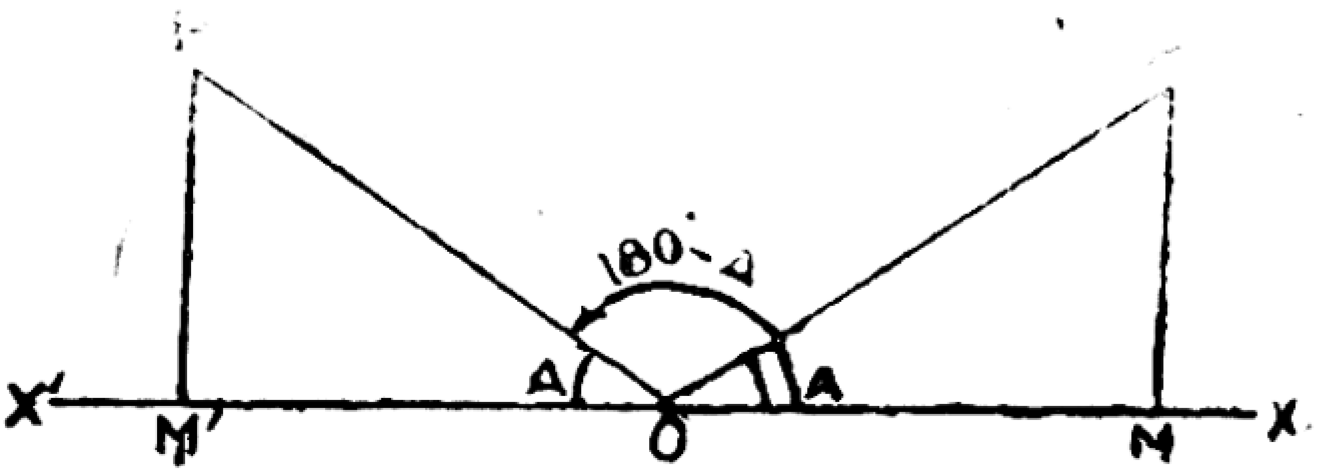
Example 1. Given sin A=cos 2A, find **A.**

sin A may be written as cos (90°$-$A),

$$\therefore \quad \cos (90°-A)=\cos 2A$$

$$\therefore \quad 90°-A=2A \text{ or } A=30°.$$

77. Def. When the sum of two angles is equal to two right angles, each is said to be the **supplement** of the other and the angles are said to be **supplementary**. Thus angles 180°$-$A



and A are supplementary and angle 180°−A is the supplement of angle A.

78. Functions of supplementary Angles, *viz.,* **A and (180°−A).** Let OP be one position of the revolving line such that ∠XOP=A, and OP′ another position such that ∠XOP′ =180°−A. (See Fig. Art. 77).

XOX′ is a straight line.

$$\therefore \angle P'OX' = 180° - \angle XOP' = 180° - (180° - A) = A.$$

Draw PM and P′M′ the ordinates of P and P′.

△s POM and P′OM′ are congruent, because OP=OP′, ∠POM=∠P′OM′, and ∠s PMO, P′M′O are rt. angles.

$$\therefore \quad MP = M'P', \text{ and } OM' = -OM.$$

$$\sin(180° - A) = \frac{M'P'}{OP'} \quad [\text{Def.}] = \frac{MP}{OP} = \sin A.$$

$$\cos(180° - A) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A.$$

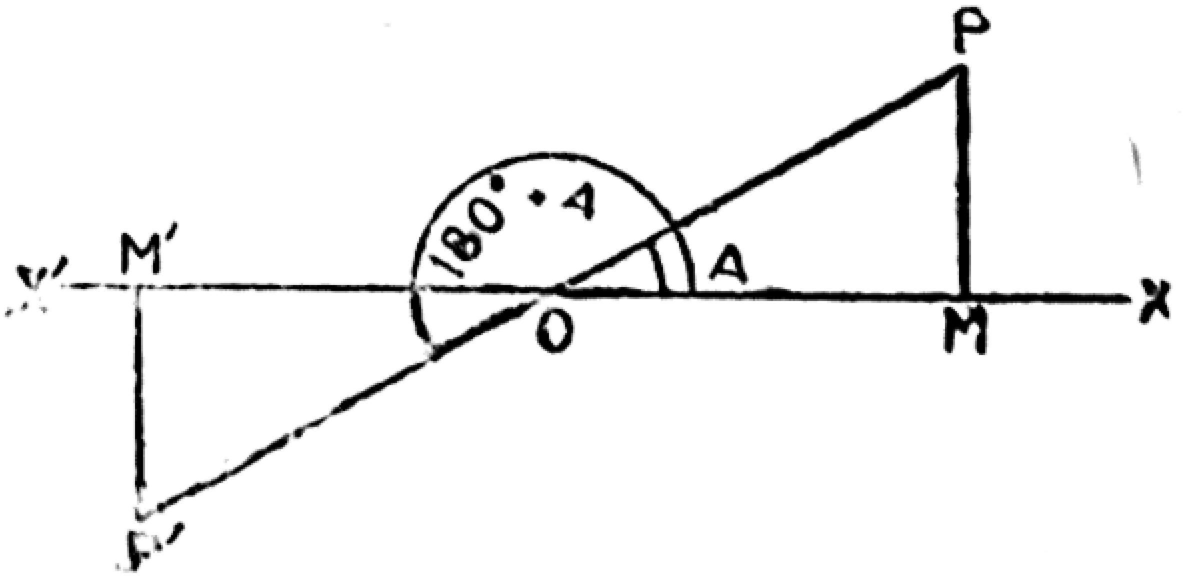
$$\tan(180° - A) = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan A.$$

Example 2. Find the sine and tangent of 135°.

$$\sin 135° = \sin(180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}},$$

$$\tan 135° = \tan(180° - 45°) = -\tan 45° = -1.$$

79. Functions of 180°+A. Let OP and OP′ be the two positions of the revolving line such that ∠XOP=A and ∠XOP′ =180°+A.



Draw PM and P′M′ ⊥ to XX′. ∠M′OP′ is also equal to A and △s POM and P′OM′ are congruent.

M′P′ = −MP and OM′ = −OM.

$$\sin(180°+A)=\frac{M'P'}{OP'}=\frac{-MP}{OP}=-\sin A.$$

$$\cos(180°+A)=\frac{OM'}{OP'}=\frac{-OM}{OP}=-\cos A.$$

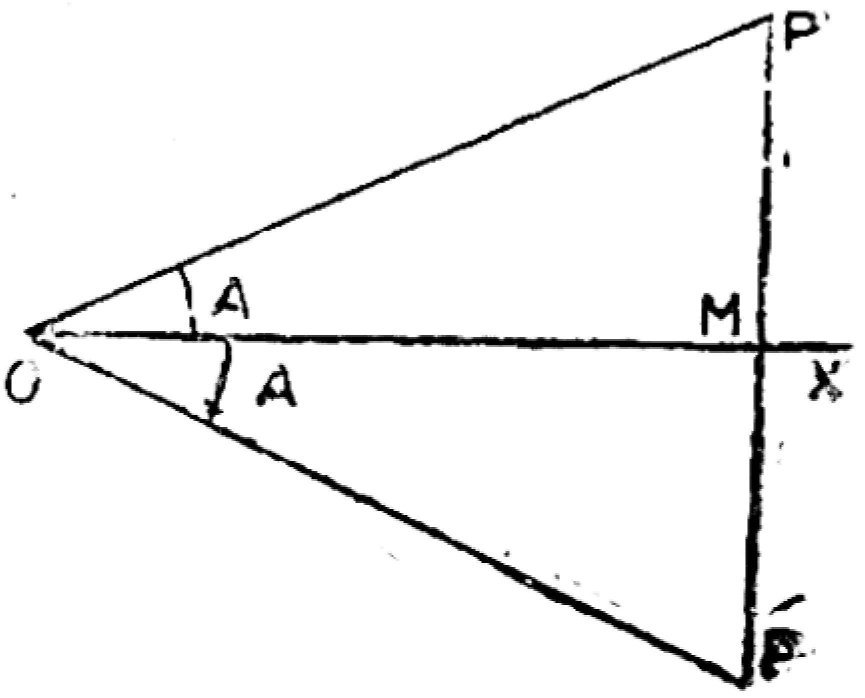
$$\tan(180°+A)=\frac{M'P'}{OM'}=\frac{-MP}{-OM}=\frac{MP}{OM}=\tan A.$$

Example 3. Find cot 240°.

$$\cot 240°=\frac{1}{\tan 240°}=\frac{1}{\tan(180°+60°)}=\frac{1}{\tan 60°}=\frac{1}{\sqrt{3}}.$$

80. Functions of −A.

Let ∠XOP = A and let the radius vector start from OX and revolve in the negative (clockwise) direction and take up the position OP′ such that angle XOP′ = −A.



Join PP′ intersecting OX at M. In the triangles POM and OP′M, OP = OP′, OM is common. ∠POM is numerically equal to ∠P′OM,

∴ △s are congruent : MP′ = −MP.

$$\sin(-A)=\frac{MP'}{OP'}=\frac{-MP}{OP}=-\sin A.$$

$$\cos(-A)=\frac{OM}{OP'}=\frac{OM}{OP}=\cos A.$$

$$\tan\ (-A)=\frac{MP'}{OM}=\frac{-MP}{OM}=-\tan A.$$

Example 4. Find sin (−225°).

$\sin\ (225°)=-\sin 225°.$

$$=-\sin\ (180°+45°)=\sin\ 45°=\frac{1}{\sqrt{2}}.$$

It will be noticed that—225° lies in the second quadrant and has thus its sine positive.

81. In order to remember all these relations; the following points should be grasped :—

(1) In relations connected with 90° *viz.*, 90°+A and 90°−A, functions without 'co' (sin, tan, sec) become functions with 'co' (cos, cot and cosec) *vice versa.*

(2) In relations containing 180° *viz.*, 180°+A and 180°−A, the functions remain as before *viz.*, sine remains as sine, cosine as cosine, etc.

(3) The sign of the function is indicated by the quadrant in which the angle lies, *e.g.*, when A is acute.

90°−A is in the first quadrant, ∴ no change of sign.

180°−A lies in the second quadrant, therefore sin remains positive and cos and tan negative.

180°+A lies in the third quadrant, therefore tan remains positive and sin and cos negative.

− A lies in the fourth quadrant, therefore cos remains positive and sin and tan negative.

82. For the sake of simplicity ∠A has been taken acute in all the above considerations. The relations are, however, true for ∠A of any magnitude.

83. Functions of 360°+A are the same as those of A because its bounding lines are coincident with those of A.

Functions of 360°−A are the same as those of −A for the same reason.

SOLVED EXAMPLES

Example 5. Find the value of cot 585°

$$\cot 585^\circ = \cot(360^\circ + 225^\circ) = \cot(225^\circ)$$
$$= \cot(180^\circ + 45^\circ) = \cot 45^\circ = 1.$$

Example 6. Prove that

$\cos A \tan A \tan(90^\circ - A) \operatorname{cosec}(90^\circ - A) = 1.$

$$\cos A \tan A \tan(90^\circ - A) \operatorname{cosec}(90^\circ - A)$$
$$= \cos A \tan A \cot A \frac{1}{\sin(90^\circ - A)}$$
$$= \cos A \tan A \frac{1}{\tan A} \times \frac{1}{\cos A} = 1.$$

Example 7. Prove that

$\tan A + \tan(180^\circ - A) + \tan(-A) = \tan(360^\circ - A).$

Left hand side $= \tan A - \tan A - \tan A$
$= -\tan A.$

Right hand side also $= \tan(360^\circ - A)$
$= \tan(-A) = -\tan A.$

Example 8. Express $\cot(A - 90^\circ)$ as function of A.

$$\cot(A - 90^\circ) = \cot\{-(90^\circ - A)\}$$
$$= -\cot(90^\circ - A) = -\tan A.$$

EXAMPLES XIV

Find the value of—

1. sin 300°. **2.** tan 855°. **3.** cot 405°.

4. sec 210°. **5.** cos 1035°.

Express as functions of A :—

6. sin (A−90°). **7.** sin (270°−A). **8.** $\tan(A - \pi)$.

Prove the following identity :—

9. $\sin(180^\circ - A)\cos(90^\circ - A) - \cos(-A)\cos(180^\circ - A) = 1.$

10. $\dfrac{\cos A}{\operatorname{cosec}(360^\circ + A)} \times \dfrac{\tan(270^\circ - A)}{\cos(-A)} = \cos A.$

11. $\dfrac{\sin(180^\circ-A)}{\tan(180^\circ+A)}\times\dfrac{\cot(90^\circ-A)}{\tan(90^\circ+A)}\times\dfrac{\cos(360^\circ-A)}{\sin(-A)}=\sin A.$

12. $\sec(360^\circ-A)+\operatorname{cosec}(720^\circ+A)=\dfrac{1}{\cos A}+\dfrac{1}{\sin A}$

Simplify

13. $\dfrac{\operatorname{cosec}(180^\circ-A)}{\sec(180^\circ+A)}\times\dfrac{\cos(-A)}{\sin(180^\circ+A)}.$

14. $\dfrac{\sin(-A)}{\sin(180^\circ+A)}-\dfrac{\cot(180^\circ-A)}{\cot A}+\dfrac{\cos A}{\sin(90^\circ-A}$

15. $\dfrac{\cos(90^\circ-A)\cos(180^\circ-A)\tan(180^\circ-A)}{\sin(90^\circ-A)\sin(180^\circ-A)\tan(360^\circ-A)}$

16. What is meant by the Circular Measure of an angle ?

A wheel of radius 12 cm. travels a distance of 122 cm. along the ground. What is the angle in degrees described by it in traversing this distance ? Find its sine. *(P.U. 1938)*

CHAPTER XV

Tables of Trigonometrical Ratios

84. The values of trigonometrical ratios of various angles between 0° and 90° have once for all been calculated by the help of certain formulae. For ready reference these values have been written down in the form of tables, so that any function of any angle may be read off directly from them.

Tables of sines, cosines and trangents are generally available to different degrees of accuracy. Usually these are :—

(*a*) **Seven-Figure Tables.**

(*b*) **Five-Figure Tables.**

(*c*) **Four-Figure Tables.**

In the Appendix, tables of sines, cosines, tangents and cotangents are given correct to four places of decimals.

Cosines can also be obtained from the sine tables as explained on *p.* 91.

SINES

85. A portion of the sine table between 40° and 44° is given as a specimen on page 90.

The left hand vertical column of the sine table gives angles in degrees from 0° to 90°. The next ten columns on the right give the values of the sines of the angles corresponding to the degrees in the left hand column and minutes (0′, 6′, 12′, etc.) at the top. The five difference columns on the right hand side of the table give the numbers to be added corresponding to 1′, 2′, 3′, etc. As in logarithmic tables, these differences are added in such a way that the unit figure of this difference corresponds to the right hand (fourth) figure of the numbers given in the preceding columns. The decimal point has been omitted as in tables of logarithms.

Example 1. Find $\sin 43^\circ 51'$ from the tables.

Opposite to 43° in the left hand column and under 48′ we have 6921. In the same horizontal line the difference for $3'=6$. Adding these we get 6927.

Thus $$\sin 43^\circ 31' = 0{\cdot}6927.$$

COSINES

86. Cosine tables, similar to sine tables are available, but they are generally a source of confusion because cosine decreases as the angle increases and ~~the numbers~~ in the difference columns are to be subtracted. In almost all the other tables, sine, tan, log, antilog, etc., these are added and as a matter of habit one is apt to add them in the case of cosines also and thus a mistake is committed.

The student is advised to find cosines from the sine tables as explained in the following example. A little extra trouble in subtracting the given angle from 90° is more than repaid by the lesser chance of mistake.

Example 1. Find $\cos 48^\circ 27'$.

$$\begin{aligned}\cos 48^\circ 27' &= \sin (90^\circ - 48^\circ 27') = \sin 41^\circ 33' \\ &= 0{\cdot}6633 \text{ from the tables.}\end{aligned}$$

NATURAL SINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°·0	0°·1	0°·2	0°·3	0°·4	0°·5	0°·6	0°·7	0°·8	0°·9	1′	2′	3′	4′	5′
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6823	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL TANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°·0	0°·1	0°·2	0°·3	0°·4	0°·5	0°·6	0°·7	0°·8	0°·9	1′	2′	3′	4′	5′
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	$\overline{3}416$	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43

TANGENTS

87. The Tangent table is similar to a sine table and is given in the appendix. The method of using it is also similar. The tangents of angles greater than 45° are greater than 1 but the integral portion is given in the 0′ column only and is omitted in the other columns to avoid repetition.

A portion of tangent tables between 50° and 54° is given as a specimen on page 90.

Example 2. Find tan 52° 20′.

Opposite to 52° and under 18′ we have 2938, which means 1·2938

Diff. for 2′ = 16 which is to be added to 2938.

$\therefore$ tan 52°20′ = 1·(2938 + 16) = 1·2954.

METHOD OF PROPORTIONAL PARTS

88. Sometimes the tables available give the sine, tangent, etc., of degrees only. For finding the function of an angle having also the fractions of a degree, the method of proportional parts illustrated below is applied.

Example 3 Given a table showing the sines of degrees only, find sin 35°·4.

From the table of sines in the appendix we find that

sin 35° = ·5736, and sin 36° = ·5878.

At this part of the table the difference in sine for 1°

= 5878 − 5736 = 142.

(It is convenient to omit decimal points in such calculations.)

Diff. for 1° = 142.

" " ·1° = 14·2

" " ·4° = 56·8 = 57 (·8 of the fifth figure has been rejected to keep four places only)

sin 35°·4 = (·5736 + 57) = ·5793.

89. The reverse process of finding the angle whose sine, cosine or tangent is given will be understood from the following example.

Example 4. Find the angle whose sine $= \cdot 6999$.

Looking up in the table (page 90) for the nearest number less than 6999, we find 6997 against 44° and under 24′. This is short of the required value by 2.

In the column of differences 2 is given under 1′.

$\therefore$ the required angle $= 44^\circ\ 25'$.

SOLVED EXAMPLES

Example 5. Find sin 250°.

$$\sin 250^\circ = \sin(180^\circ + 70^\circ) = -\sin 70^\circ$$
$$= -\cdot 9397 \text{ (from the tables).}$$

Example 6. Find cos 290°.

$$\cos 290^\circ = \cos(360^\circ - 70^\circ) = \cos 70^\circ$$
$$= \cos(90^\circ - 20^\circ) = \sin 20^\circ = \cdot 3420.$$

Example 7. Find tan 920°.

$$\tan 920^\circ = \tan(2 \times 360^\circ + 200^\circ) = \tan 200^\circ$$
$$= \tan(180^\circ + 20^\circ) = \tan 20^\circ = \cdot 3640.$$

Example 8. Find cot 22° 27′.

$$\cot 22^\circ\ 27' = \tan(90^\circ - 22^\circ\ 27')$$
$$= \tan 67^\circ\ 33' = 2\cdot 4202.$$

Example 9. Find the angle whose cosine $= \cdot 8254$.

Angle whose sine is $\cdot 8254 = 55^\circ\ 38'$

$\therefore$ the angle whose cosine is $\cdot 8254 = 90^\circ - 55^\circ\ 38' = 34^\circ\ 22'$.

Example 10. Find the angle less than 180° whose tangent $= -2\cdot 4794$.

Angle whose tangent is $2\cdot 4794 = 68^\circ\ 2'$.

$\therefore$ the angle whose tan is $-2\cdot 4794 = 180^\circ - 68^\circ\ 2' = 111^\circ\ 58'$.

EXAMPLES XV

[Use of tables is necessary]

1. Verify the value of sines, cosines and tangents of 0°, 30°, 45°, 60° and 90°.

Find the value of

2. $\sin 50^\circ 27'$. **3.** $\cos 15^\circ 44'$. **4.** $\tan 43^\circ 20'$.

5. $\cot 24^\circ 31'$. **6.** $\tan 185^\circ{\cdot}2$. **7.** $\cos 133^\circ{\cdot}3$.

8. $\sin 293^\circ{\cdot}4$.

Find angles less than 180° whose

9. sine $= {\cdot}2345$. **10.** cosine $= {\cdot}7254$.

11. tangent $= 3{\cdot}2138$. **12.** cotangent $= {\cdot}1237$.

13. cosine $= -{\cdot}3215$. **14.** tangent $= -1{\cdot}3421$.

CHAPTER XVI

Addition and Subtraction Formulæ

90. *To prove that* :—

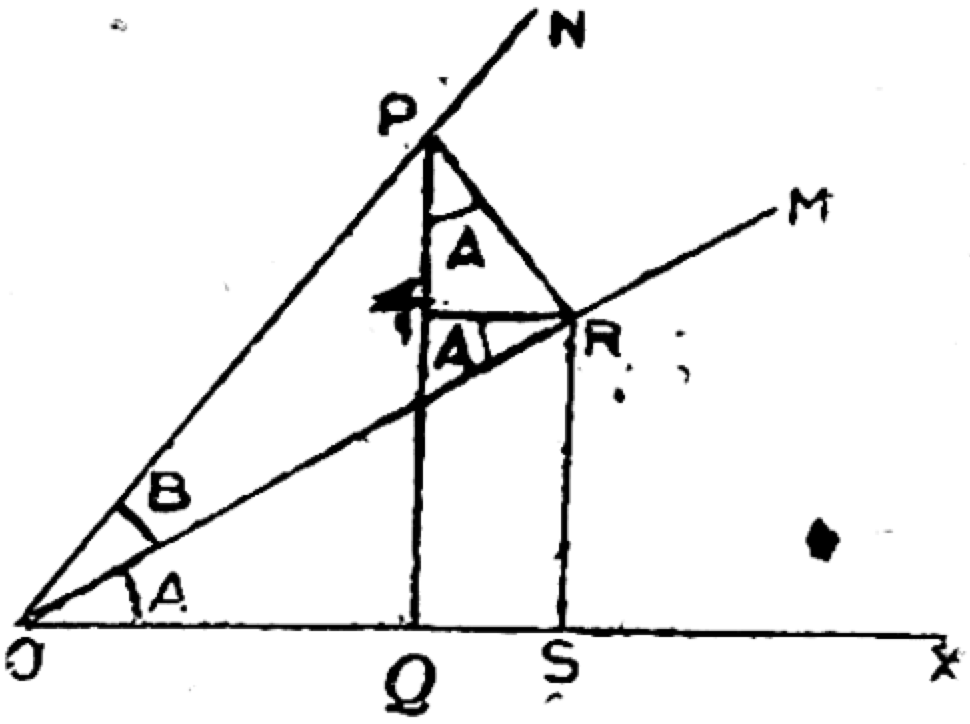
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Let $\angle XOM = A$.

and $\angle MON = B$.

$$\angle XON = \angle XOM + \angle MON$$
$$= A + B.$$



Take any point P on ON, the boundary line of A+B. Draw PQ and PR perpendiculars to OX and OM respectively, and also draw RS and RT perpendiculars to OX and PQ respectively.

$\angle RPT = 90^\circ - \angle TRP = \angle TRO = A$ [TR and OX being ||]

$$\sin(A+B) = \frac{PQ}{OP} = \frac{TQ+PT}{OP}$$

$$= \frac{RS+PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP}$$

$$= \frac{RS}{OR} \times \frac{OR}{OP} + \frac{PT}{PR} \times \frac{PR}{OP}$$

$$=\sin A\cos B+\cos A\sin B$$

$$\cos(A+B)=\frac{OQ}{OP}=\frac{OS-QS}{OP}$$

$$=\frac{OS-TR}{OP}=\frac{OS}{OP}-\frac{TR}{OP}$$

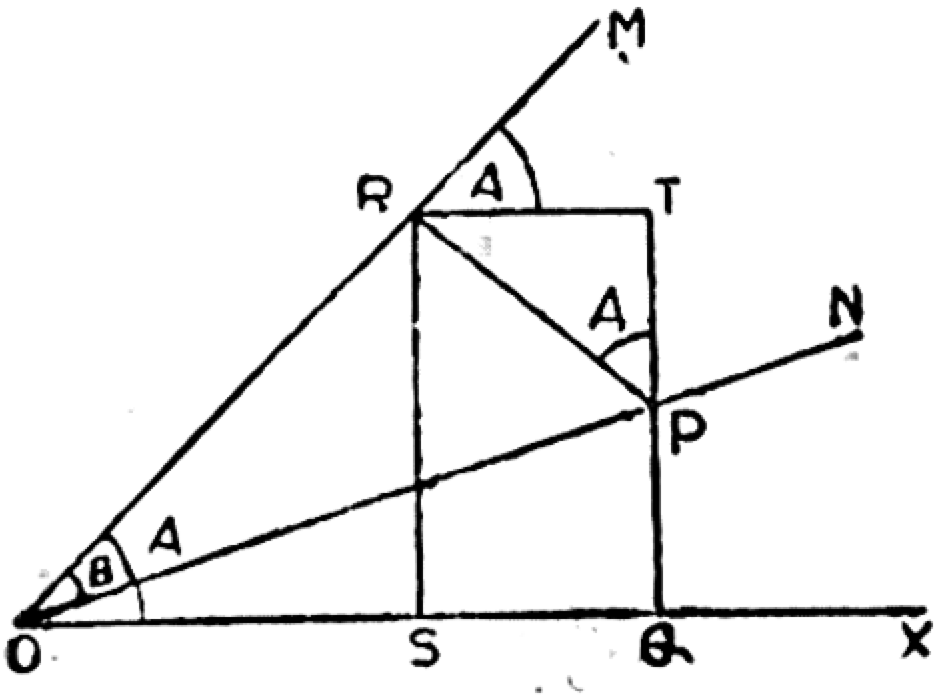
$$=\frac{OS}{OR}\times\frac{OR}{OP}-\frac{TR}{PR}\times\frac{PR}{OP}$$

$$=\cos A\cos B-\sin A\sin B.$$

91. *To prove that*

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$



As before let

$\angle XOM=A$ and $\angle NOM=B$.

$\angle XON=\angle XOM-\angle NOM$

$=A-B$.

Take a point P on ON, the boundary line of A−B.

Draw PQ and PR perpendiculars to OX and OM respectively and also draw RS and RT perpendiculars to OX and PQ respectively.

$\angle TPR=90^\circ-\angle PRT=\angle TRM=A$. [$\because$ RT is $\parallel$ to OX]

$$\sin(A-B)=\frac{PQ}{OP}=\frac{TQ-PT}{OP}$$

$$=\frac{RS-PT}{OP}=\frac{RS}{OP}-\frac{PT}{OP}$$

$$=\frac{RS}{OR}\times\frac{OR}{OP}-\frac{PT}{PR}\times\frac{PR}{OP}$$

$$=\sin A\cos B-\cos A\sin B.$$

$$\cos(A-B)=\frac{OQ}{OP}=\frac{OS+SQ}{OP}$$

$$=\frac{OS+RT}{OP}=\frac{OS}{OP}+\frac{RT}{OP}$$

$$=\frac{OS}{OR}\times\frac{OR}{OP}+\frac{RT}{PR}\times\frac{PR}{OP}$$

$$=\cos A\cos B+\sin A\sin B.$$

92. *To prove that :—*

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B)=\frac{\tan A-\tan B.}{1+\tan A\tan B}$$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

Divide both the numerator and denominator by cos A cos B.

$$\text{Thus } \tan(A+B)=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\times\dfrac{\sin B}{\cos B}}=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B.}$$

Divide both the numerator and denominator by cos A cos B.

$$\text{Thus } \tan(A-B)=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A}{\cos A}\times\dfrac{\sin B}{\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

Caution. *It should be clearly understood that the sine of the sum of two angles is not equal to the sum of their sines, viz.,* **sin (A+B)** *is not equal to* **sin A+sin B**. *Similarly the others.*

93. *To prove that :—*

(*i*) $\sin 2A = 2 \sin A \cos A.$

(*ii*) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$

*(*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}.$

*(*iv*) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

*(*v*) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}.$

(*i*) $\sin 2A = \sin (A + A)$

$= \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A.$

(*ii*) $\cos 2A = \cos (A + A) = \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$

$= 2 \cos^2 A - 1.$

and is also $= (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A.$

*(*iii*) $\tan 2A = \tan (A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A} = \dfrac{2 \tan A}{1 - \tan^2 A}.$

*(*iv*) $\sin 2A = 2 \sin A \cos A$ [by (*i*)

$= \dfrac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$ [$\cos^2 A + \sin^2 A = 1$

$= \dfrac{\dfrac{2 \sin A}{\cos A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}}$ [dividing numerator and deneminator by $\cos^2 A$.

$= \dfrac{2 \tan A}{1 + \tan^2 A}$

*(*v*) $\cos 2A = \cos^2 A - \sin^2 A$ [by (*ii*)

$= \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$ [$\cos^2 A + \sin^2 A = 1$

$$=\frac{1-\dfrac{\sin^2 A}{\cos^2 A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}$$ [dividing numerator and denominator by $\cos^2 A$

$$=\frac{1-\tan^2 A}{1+\tan^2 A}.$$

Caution. *sin 2A is not equal to 2 sin A, and so on.*

94. *To find the circular functions of 15°.*

$$\sin 15^\circ=\sin(60^\circ-45^\circ)$$
$$=\sin 60^\circ \cos 45^\circ-\cos 60^\circ \sin 45^\circ$$
$$=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos 15^\circ=\cos(60^\circ-45^\circ)$$
$$=\cos 60^\circ \cos 45^\circ+\sin 60^\circ \sin 45^\circ$$
$$=\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$^*\tan 15^\circ=\tan(60^\circ-45^\circ)$$
$$=\frac{\tan 60^\circ-\tan 45^\circ}{1+\tan 60^\circ \tan 45^\circ}=\frac{\sqrt{3}-1}{1+\sqrt{3}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$
$$=\frac{(\sqrt{3}-1)^2}{3-1}=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}.$$

Otherwise

$$\tan 15^\circ=\frac{\sin 15^\circ}{\cos 15^\circ}=\frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{2\sqrt{2}}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}.$$

SOLVED EXAMPLES

Example 1. Prove that

$$\cos 2A \cos A+\sin 2A \sin A=\cos A.$$

The very form of the left-hand expression shows that it is the expansion of $\cos(2A-A)=\cos A$.

Otherwise

$$\begin{aligned}\cos 2A\cos A+\sin 2A\sin A &= (1-2\sin^2 A)\cos A+2\sin A\cos A\sin A\\ &=\cos A-2\cos A\sin^2 A+2\cos A\sin^2 A\\ &=\cos A.\end{aligned}$$

Example 2. Given $\cos 2A = {\cdot}96$, find $\tan A$.

$$\cos 2A = {\cdot}96,$$

$$\therefore\quad 2\cos^2 A - 1 = {\cdot}96$$

i.e.,

$$\cos^2 A = \frac{1{\cdot}96}{2} = {\cdot}98.$$

$\cos 2A$ is also equal to $1-2\sin^2 A$

$$\therefore\quad 1-2\sin^2 A = {\cdot}96.$$

$$\therefore\quad \sin^2 A = \frac{1-{\cdot}96}{2} = {\cdot}02.$$

$$\tan A = \sqrt{\frac{\sin^2 A}{\cos^2 A}} = \sqrt{\frac{{\cdot}02}{{\cdot}98}}$$

$$= \sqrt{\frac{1}{49}} = \frac{1}{7} = {\cdot}1429.$$

Aliter.

$$\cos 2A = {\cdot}96$$

$$\therefore\quad \frac{1-\tan^2 A}{1+\tan^2 A} = {\cdot}96$$

$$1-\tan^2 A = {\cdot}96 + {\cdot}96\tan^2 A$$

$$-1{\cdot}96\tan^2 A = {\cdot}96 - 1 = -{\cdot}04$$

or

$$\tan^2 A = \frac{-{\cdot}04}{-1{\cdot}96} = \frac{1}{49}$$

or

$$\tan A = \sqrt{\frac{1}{49}} = \frac{1}{7} = {\cdot}1429.$$

Example 3. Prove that $\dfrac{1+\sin A-\cos A}{1+\sin A+\cos A} = \tan\dfrac{A}{2}.$

We have $\sin 2\theta = 2\sin\theta\cos\theta.$

Let $2\theta = A$, then $\theta = \dfrac{A}{2}.$

$$\therefore\quad \sin A=2\sin\frac{A}{2}\cos\frac{A}{2}.$$

Similarly $\cos A=2\cos^2\frac{A}{2}-1$, etc.

$$\frac{1+\sin A-\cos A}{1+\sin A+\cos A}=\frac{1+2\sin\frac{A}{2}\cos\frac{A}{2}-\left(1-2\sin^2\frac{A}{2}\right)}{1+2\sin\frac{A}{2}\cos\frac{A}{2}+\left(2\cos^2\frac{A}{2}-1\right)}$$

$$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}+2\cos^2\frac{A}{2}}$$

$$=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\left\{\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\sin\frac{A}{2}+\cos\frac{A}{2}}\right\}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\tan\frac{A}{2}.$$

Example 4. If $A+B+C=180°$, prove that

$\tan A+\tan B+\tan C=\tan A, \tan B, \tan C.$ *(P.U. 1945)*

$$A+B+C=180°.$$

$$A+B=180°-C.$$

$$\tan(A+B)=\tan(180°-C).$$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C.$$

Multiplying crosswise

$$\tan A+\tan B=-\tan C+\tan A\tan B\tan C.$$

$$\therefore\ \tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Example 5. Prove that

$\tan 51°-\tan 31°-\tan 20°=\tan 51°\tan 31°\tan 20°$

$$\tan 51°=\tan(31°+20°)$$

$$=\frac{\tan 31°+\tan 20°}{1-\tan 31°\tan 20°}$$

Multiplying crosswise

$\tan 51^\circ - \tan 51^\circ \tan 31^\circ \tan 20^\circ = \tan 31^\circ + \tan 20^\circ$.

$\therefore \quad \tan 51^\circ - \tan 31^\circ - \tan 20^\circ = \tan 51^\circ \tan 31^\circ \tan 20^\circ$

Example 6. Prove that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$

$$\sin 105^\circ + \cos 105^\circ = \sin(60^\circ + 45^\circ) + \cos(60^\circ + 45^\circ)$$
$$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ + \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ.$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Example 7. Find the circular functions of $22\frac{1}{2}^\circ$.

$$\cos 45^\circ = 1 - 2\sin^2 22\tfrac{1}{2}^\circ.$$

$$\sin^2 22\tfrac{1}{2}^\circ = \frac{1 - \cos 45^\circ}{2} = \frac{1 - \frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2} - 1}{2\sqrt{2}},$$

$$\therefore \quad \sin 22\tfrac{1}{2}^\circ = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}$$

$$\cos 45^\circ = 2\cos^2 22\tfrac{1}{2}^\circ - 1.$$

$$\therefore \quad \cos^2 22\tfrac{1}{2}^\circ = \frac{1 + \cos 45^\circ}{2} = \frac{1 + \frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2} + 1}{2\sqrt{2}}$$

$$\therefore \quad \cos 22\tfrac{1}{2}^\circ = \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$$

$$\tan 22\tfrac{1}{2}^\circ = \frac{\sin 22\frac{1}{2}^\circ}{\cos 22\frac{1}{2}^\circ} = \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}}$$

EXAMPLES XVI

1. Prove that $2\sin(30^\circ - A) = \cos A - \sqrt{3} \sin A$.
2. If $\sin A = \frac{5}{13}$ and $\sin B = \frac{3}{5}$, find $\cos(A + B)$.

3. If cosec $A=\frac{61}{60}$ and cosec $B=\frac{41}{40}$; find $\sin(A+B)$ and $\cos(A-B)$.

4. If $\cos A=\frac{8}{17}$ and $\sin B=\frac{4}{5}$, find $\sec(A+B)$.

5. If $\tan 2\theta=\frac{3}{4}$, find $\tan\theta$.

6. If $\tan A=\dfrac{120}{119}$, find $\sin\dfrac{A}{2}$ and $\cos\dfrac{A}{2}$.

7. If $\tan A=\frac{4}{3}$ and $B=45^\circ$, find $\tan(A-B)$.

8. Find the circular functions of 75°.

Prove the following identities—

9. $\dfrac{\cos(A-B)}{\cos A\sin B}=\tan A+\cot B$.

10. $\sin(2A-B)\cos(2B-A)+\cos(2A-B)\sin(2B-A)$
$=\sin(A+B)$.

11. $\cos(2A-B)\cos(2B-A)+\sin(2A-B)\sin(2B-A)$
$=\cos 3(A-B)$.

12. $\sin(A+B)\sin(A-B)=\sin^2A-\sin^2A$.

13. $\cos(A+B)\cos(A-B)=\cos^2A-\sin^2B$.

14. $\cos(45^\circ+A)+\sin(A-45^\circ)=0$.

*15. $\tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}$. *16. $\tan(45^\circ-A)=\dfrac{1-\tan A}{1+\tan A}$.

17. $\tan A-\tan\dfrac{A}{2}=\tan\dfrac{A}{2}\sec A$.

*18. $\tan A+\cot 2A=\operatorname{cosec} 2A$.

19. $\cos(A-B)\cos(A+B)-\sin(A-B)\sin(A+B)=\cos 2A$.

*20. $\tan 2A\tan 3A\tan 5A=\tan 5A-\tan 3A-\tan 2A$.

21. $\tan 63^\circ\tan 43^\circ\tan 20^\circ=\tan 63^\circ-\tan 43^\circ-\tan 20^\circ$.

*22. $$\frac{\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)}=\sin 2A.$$

23. $(\cos A+\cos B)(\cos A-\cos B)=\frac{1}{2}(\cos 2A-\cos 2B)$.

24. $\sin 8A=8\sin A\cos A\cos 2A\cos 4A$.

25. Show that $\cos(A+B)=\cos A \cos B-\sin A \sin B$. *(P.U. 1929, 1940)*

26. Show that $\sin(A+B)=\sin A \cos B+\cos A \sin B$. *(P.U. 1930)*

CHAPTER XVII

Solution of Right-Angled Triangles

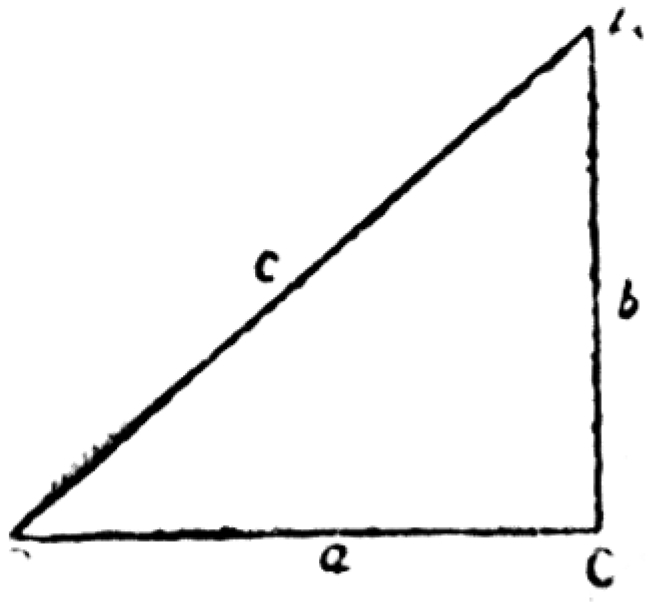
95. The science of trigonometry was invented for the purpose of solution of triangles *viz.*, the determination of the unknown sides and angles of a triangle from certain known ones. The three sides and the three angles of a triangle are called its six **parts**, and when any three of these are given, provided that, one at least is a side, the values of the others can be calculated. The solution of **right-angled triangles** only will be considered in this chapter.

96. One angle of a right-angled triangle is equal to 90° and is thus always known. If any two of the other parts are given one at least being a side, the triangle can be solved. Unless specially mentioned otherwise, the word side means any of the three sides of the right-angled triangle. Only two cases arise—

(1) When any two sides are given.

(2) When one side and one acute angle are given.

97. *To solve a right-angled triangle when two sides are given.*



Let ABC be a right-angled triangle with angle C equal to a right angle.

Angles are generally denoted by the capital letters A, B and C and the sides by the small letters a, b and c, corresponding to the opposite angles.

We have $c^2=b^2+a^2$.

Thus if any two of the sides are given the third can be calculated.

$$\sin B=\frac{b}{c},\ \cos B=\frac{a}{c} \text{ and } \tan B=\frac{b}{a}.$$

If any two of the sides are given, one of these three relations will exist beween them, thus the value of B can be found with the help of the talbes if necessary.

Having found B, we can find A from the relation $A+B=90°$. Thus all the sides and the angles of the triangle are known.

Example 1. Given $C=90°$, $b=12$ and $c=13$. Solve the triangle.

(See the figure of the last article.)

$$a=\sqrt{c^2-b^2}=\sqrt{13^2-12^2}=\sqrt{169-144}=\sqrt{25}=5.$$

$$\sin B=\tfrac{12}{13}=\cdot 9231,$$

$$\therefore \qquad B=67° \, 23' \text{ from the tables.}$$

$$A=90°-67° \, 23'=22° \, 37'$$

Example 2. $C=90°$, $b=12$, $a=4\sqrt{3}$; solve the triangle.

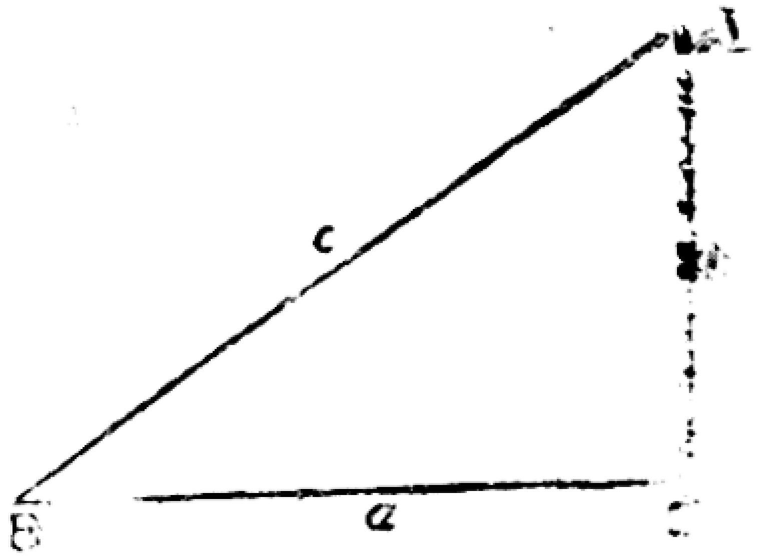
(Use the same figure as in the last article.)

$$c=\sqrt{a^2+b^2}=\sqrt{144+48}=\sqrt{192}=8\sqrt{3}.$$

$$\tan B=\frac{12}{4\sqrt{3}}=\sqrt{3}, \quad \therefore\ B=60°.$$

$$\text{Hence } A=90°-60°=30°.$$

98. *To solve a right-angled triangle when one side and one acute angle are given.*



Let ABC be a right-angled triangle with angle $C=90°$.

If one of the acute angles A or B is given the other can be found from the relation $A+B=90°$.

We have $\sin B=\dfrac{b}{c}$, $\cos B=\dfrac{a}{c}$ and $\tan B=\dfrac{b}{a}$.

If c is given, first and second of these relations will yield the values of b and c.

If a is given, second and third of these relations will yield the values of b and c.

If b is given, first and third of these relations will yield the values of c and a.

Example 3. A=30°, C=90°, $b=20\sqrt{3}$; solve the triangle.

$$B=90^\circ-30^\circ=60^\circ.$$

$$\cos A=\frac{b}{a}, \quad \therefore\ c=\frac{b}{\cos A}, \quad i.e.,\ c=\frac{20\sqrt{3}}{\frac{\sqrt{3}}{2}}=40.$$

$$\tan A=\frac{a}{b} \quad \therefore\ a=b\tan A,$$

i.e.

$$a=20\sqrt{3}\frac{1}{\sqrt{3}}=20.$$

Example 4. C=90°, A=38° 19′, $c=50$, find a.

$$\sin A=\frac{a}{c}\ ; \therefore\ a=c\sin A$$

i.e.,

$$a=50\times\cdot62 \text{ (from the tables)}=31.$$

99. In a right-angled triangle ABC like that of the figure of the last article,

$$\frac{a}{c}=\cos B,\ \therefore\ a=c\cos B\ldots\ldots\ldots(i)$$

$$\frac{b}{c}=\sin B,\ \therefore\ b=c\sin B\ldots\ldots\ldots(ii)$$

$$\frac{b}{a}=\tan B.\ \therefore\ b=a\tan B\ldots\ldots\ (iii)$$

Similarly,

$$b=c\cos A\ldots\ldots\ldots(iv)$$

$$a=c\sin A\ldots\ldots\ldots(v)$$

$$a=b\tan A\ldots\ldots\ldots(vi)$$

These relations are frequently needed and it is highly desirable that the student should be able to write them down at once from a figure.

Put in words they are .—

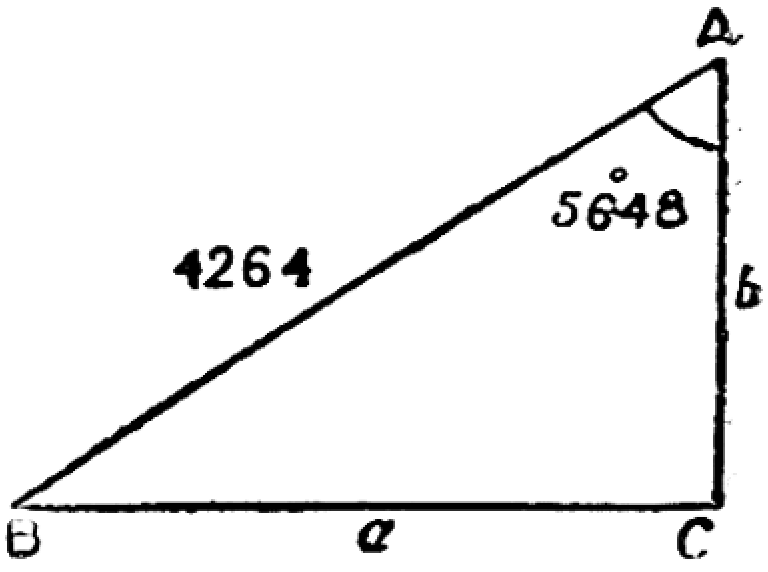
Any of the two sides of a right-angled triangle containing the right angle is given by (i) the hypotenuse multiplied by the sine of the opposite angle or the cosine of the adjacent acute angle, or (ii) the other side multiplied by the tangent of the opposite angle.

SOLVED EXAMPLES

Example 5. $C=90^\circ$, $A=56^\circ{\cdot}48$,

$c=4264$. Find a.

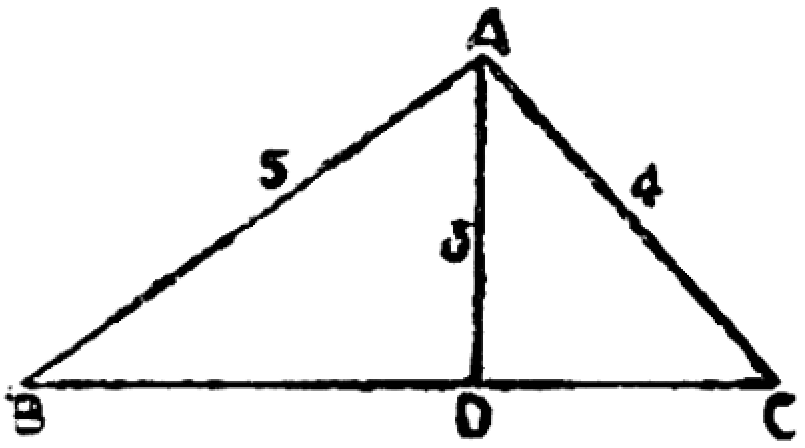
$a=c \sin A=4264 \times {\cdot}8337$.



$\log a=3{\cdot}6298+\bar{1}{\cdot}9210=3{\cdot}5508$.

$\therefore\ a=3555$.

Example 6. The length of the perpendicular from one angle of a triangle to the opposite side is equal to 3 and the lengths of the sides containing the angle are 4 and 5. Find the angles and the third side of the triangle.



Let ABC be a triangle with sides AB and AC equal to 5 and 4 respectively. $AD=3$.

$$\sin C=\frac{3}{4}={\cdot}75,$$

$\therefore$ $C=48^\circ\ 35'$ (from the tables).

$\sin B=\dfrac{3}{5}={\cdot}60$. $\therefore$ $B=36^\circ\ 52'$ (from the tables).

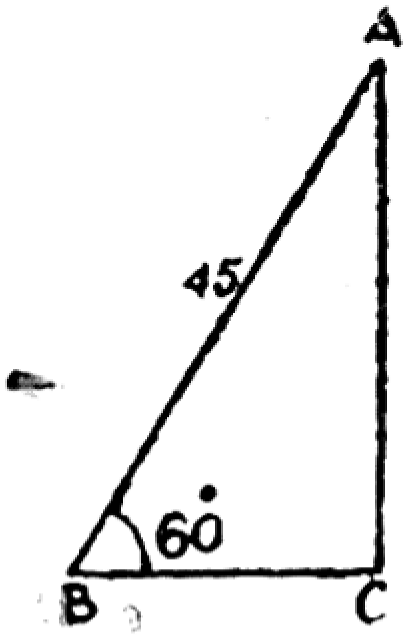
$A+B+C=180^\circ$,

$$A=180^\circ-(B+C)=180^\circ-85^\circ27'=94^\circ33'.$$

$$BD=\sqrt{5^2-3^2}=\sqrt{16}=4.$$

$$DC=\sqrt{4^2-3^2}=\sqrt{7}=2{\cdot}645.$$

$$BC=BD+DC=6{\cdot}645.$$



Example 7. A ladder 45 ft. long just reaches the top of a wall. If the ladder makes an angle of 60° with the ground, find the height of the wall and the distance of the foot of the ladder from the wall.

If AC is the wall and AB the ladder,

$$AC = AB \sin 60^\circ = 45 \times \frac{\sqrt{3}}{2} = 38{\cdot}97 \text{ ft}$$

$$BC = AB \cos 60^\circ = 45 \times \tfrac{1}{2} = 22{\cdot}50 \text{ ft.}$$

HEIGHTS AND DISTANCES

100. We cannot find the height or the distance of an inaccessible object by direct measurement. For instance, it is not possible to put one end of a measuring tape at the very top of a high tree or the peak of a mountain where we cannot climb up. Trigonometry helps us in finding the heights of such places by measuring certain angles from two or more places from which the object is visible. The measurement of angles is done by a theodolite or a sextant and the principle of solution of triangles is made use of for calculations. Distances of stars are also measured on a similar principle.

101. Let an observer stand at O and see an object Q. Draw OP a horizontal line in the same vertical plane as OQ.

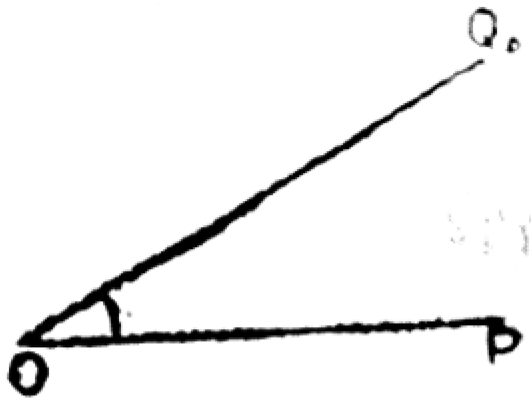


Fig. 1.

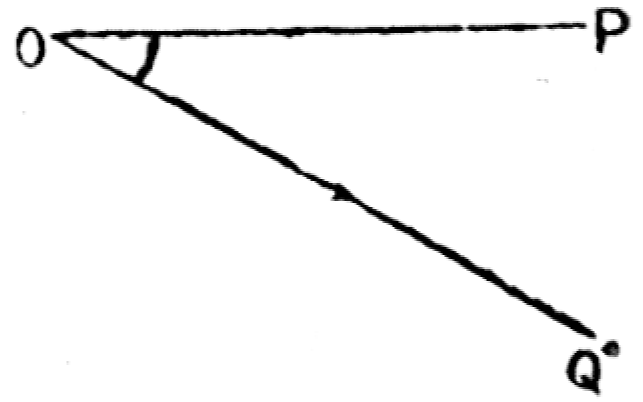


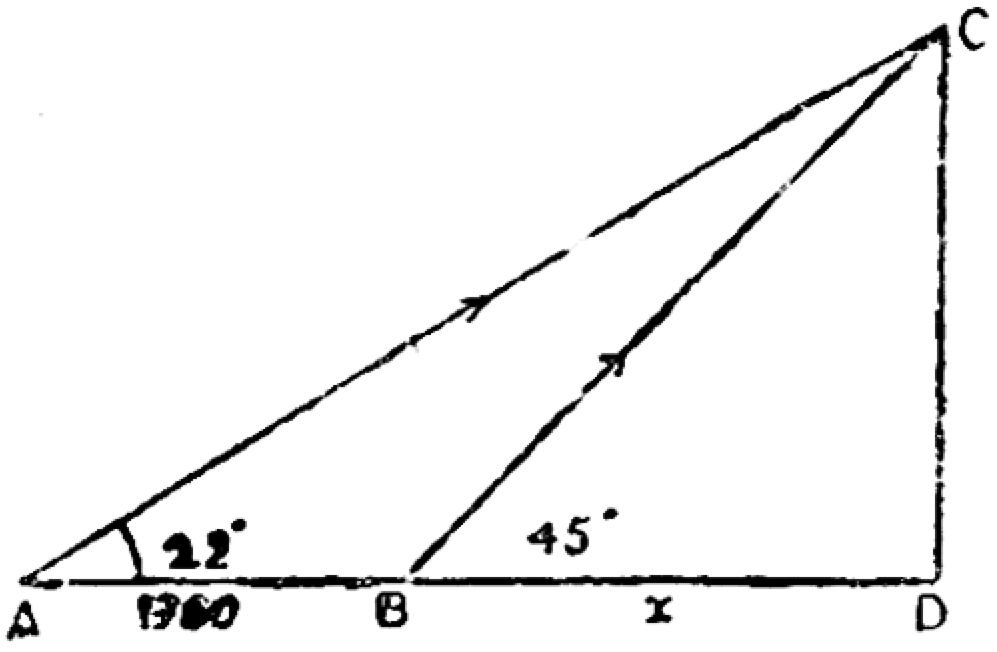
Fig. 2.

In Fig. 1, where the object is above the horizontal line OP, the angle QOP is called the **angle of elevation** of the object Q.

In Fig. 2. where the object is below the horizontal line OP, the angle QOP is called the **angle of depression** of Q.

[**Note.** In each case the angle is measured between the line of vision and the horizontal line.]

Example 8. The angles of elevation of the top of a hill from two consecutive milestones on a straight level road going towards the hill are 22° and 45° respectively. Find the height of the hill above the road.



Let A and B be the two milestones and C the top of the hill. Draw CD, vertical from C to the line AB produced. Angles CAD and CDB are 22° and 45° respectively. Let BD be equal to x.

From the △ CAD,

$$CD = AD \tan 22°,$$

$$\therefore \quad AD = \frac{CD}{\tan 22°}$$

or $$x + 1760 = CD \cot 22° \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

From the △ CBD,

$$CD = BD \tan 45°,$$

$$\therefore \quad BD = \frac{CD}{\tan 45°}$$

or $$x = BD \cot 45° \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Subtracting (ii) from (i),

$$1760 = CD\,(\cot 22° - \cot 45°).$$

$$\therefore \quad CD = \frac{1760}{\cot 22° - \cot 45°}$$

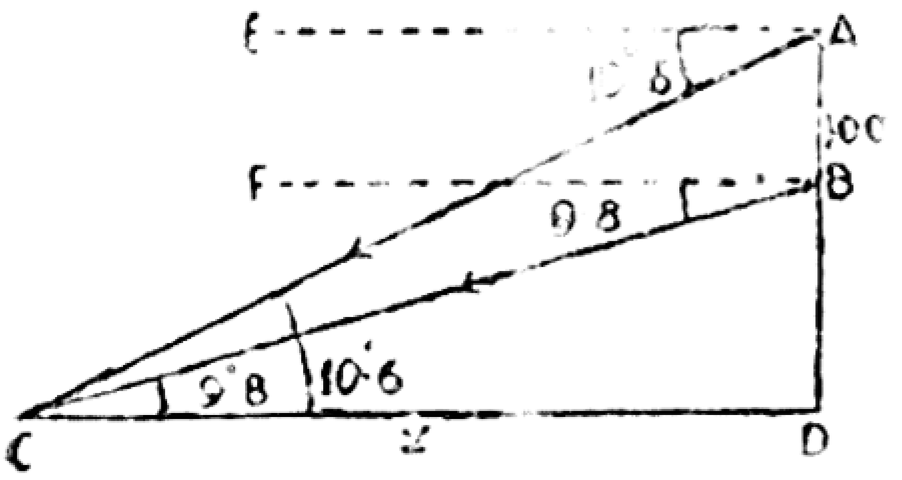
$$= \frac{1760}{2{\cdot}4751 - 1} = \frac{1760}{1{\cdot}4751}.$$

$$\log CD = 3{\cdot}2455 - {\cdot}1688 = 3{\cdot}0767,$$

$$\therefore \quad CD = 1193 \text{ yds.}$$

Any one of the distances CD, BD, AB and AD can be calculated if the two angles CAD and CBD and one of these distances are given.

Example 9. A person standing at the top of the vertical mast of a ship measures the angle of depression of a point on the shore to be 10°·6, and coming down a hundred feet, he finds, the angle of depression of the same point to be 9°·8. Find the distance of the ship from the shore.



Let A be the top of the mast and B a point 100 ft. below. If C is the point on the shore and AE and BF are horizontal lines ∠CAE $=10^\circ\cdot6$ and ∠CBF$=9^\circ\cdot8$.

Draw CD a horizontal line to meet the vertical line AB produced at D. Let CD be equal to x.

∠ACD is also equal to $10^\circ\cdot6$ and ∠BCD $=9^\circ\cdot8$.

In the △ ACD,

$$AD=x\tan 10^\circ\cdot6.$$

$$\therefore \quad 100+BD=x\tan 10^\circ\cdot6 \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

In the △ BCD,

$$BD=x\tan 9^\circ\cdot8 \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Subtracting (*ii*) from (*i*) we have

$$100=x\,(\tan 10^\circ\cdot6-\tan 9^\circ\cdot8),$$

$$\therefore \quad x=\frac{100}{\tan 10^\circ\cdot6-\tan 9^\circ\cdot8}=\frac{100}{\cdot1871-\cdot1727}=\frac{100}{\cdot0144}$$

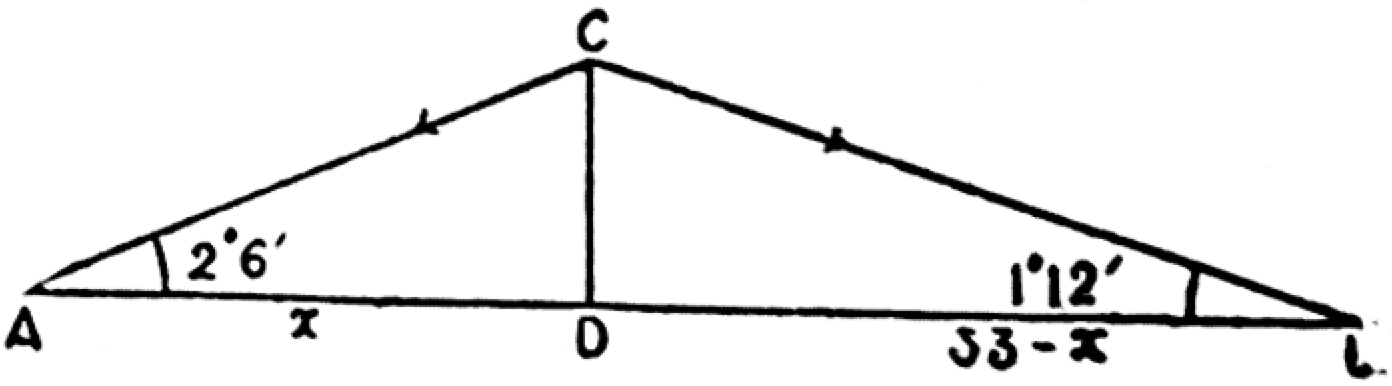
$$\log x=2-\bar{2}\cdot1584=3\cdot8416.$$

$$\therefore \quad x=6944 \text{ ft.}=1 \text{ mile } 554 \text{ yds. } 2 \text{ ft.}$$

Example 10. A person in an aeroplane flying between Lahore and Amritsar (distance 33 miles) measures their angles of depression to be 1° 12′ and 2°6′ respectively in the same vertical plane. Find the height above the ground and his distance from Lahore, assuming the line connecting *Lahore* and *Amritsar* to be horizontal.

Let A, L, and C denote Amritsar, Lahore and the Aeroplane respectively.

Draw CD ⊥ to AL.



AL is horizontal, ∴ ∠CAD$=2^\circ6'$ and ∠CLD$=1^\circ12'$ as in the previous example. Let AD$=x$ (miles) ; DL$=33-x$ miles.

In the $\triangle$ CLD.

$$CD=(33-x)\tan 1^{\circ}12',$$

$$\therefore\quad 33-x=CD\cot 1^{\circ}12' \qquad \ldots \qquad \ldots \qquad \ldots \qquad (i)$$

In the $\triangle$ CAD,

$$CD=x\tan 2^{\circ}6'$$

$$\therefore\quad x=CD\cot 2^{\circ}6' \qquad \ldots \qquad \ldots \qquad \ldots \qquad (ii)$$

Adding (i) and (ii)

$$33=CD\,(\cot 1^{\circ}12'-\cot 2^{\circ}6'),$$

$$\therefore\quad CD=\frac{33}{47{\cdot}74+27{\cdot}27}\text{ miles}=\frac{33\times1760\times3}{75{\cdot}01}\text{ ft.}$$

$$=2322\text{ ft.}$$

In the $\triangle$ CLD,

$$CD=CL\sin 1^{\circ}12',$$

$$\therefore\quad CL=\frac{CD}{\sin 1^{\circ}12'}=\frac{33}{75{\cdot}01\times{\cdot}0209}=21{\cdot}05\text{ miles.}$$

EXAMPLES XVII

Use logarithms wherever convenient and tables where needed. $\angle C=90^{\circ}$, solve the following triangles.

1. $c=4,\ b=2\sqrt{3}$. **2.** $b=12,\quad a=4$.

3. $c=20\sqrt{2},\ b=20$. **4** $A=60^{\circ},\quad c=8$.

5. $A=45^{\circ},\ b=4$. **6.** $A=54^{\circ},\quad c=9$.

7. $R=39^{\circ}{\cdot}4,\ b=25$. **8.** $a=225,\ b=272$.

9. $A=20^{\circ}14',\ b=4930$. **10.** $a={\cdot}04,\ A=40^{\circ}$.

11. $c=726{\cdot}9,\ b=316{\cdot}2$. **12.** $c=6{\cdot}953,\ b=3$.

13. $B=4^{\circ}\,73,\ b=694{\cdot}7$

14. If AD is perpendicular to the base BC of triangle ABC, find b and c; given $B=30^{\circ}$, $C=45^{\circ}$ and $AD=10$.

15. The elevation of the top of a tower from a distance 100 ft. from its foot is 23°10′; find its height.

16. The shadow of a man 5 ft. 8 in. high is 2 ft. long. Find the altitude of the sun.

17. Angle of elevation of the top of a tower 132 ft. high is 41°18′. Find the distance of the observer from the middle of the tower.

18. The angles of elevation of the top of a tower from two points 1000 ft. apart in a horizontal line passing through its base are 20° and 50°. Find the height of the tower.

19. From the foot of a post 30 ft. high the angle of elevation of a turret of a steeple is 45° and from its top the angle of elevation of the same point is 30°. Find the height of the turret above the ground and the distance of the steeple from the observer.

20. The angle of depression of an object from the top of a tower 100 ft. high is 42°15′. Find the distance of the object from the observer.

21. From the top of a cliff 150 ft. high the angles of depression of two boats both due south of the observer are 30° and 15°. How far are the boats apart ?

22. An aeroplane is flying 200 ft. high and it takes half a minute for the angle of depression of a point on the ground to change from 20°15′ to 2°12′. Find the velocity of the aeroplane.

CHAPTER XVIII

Simple Relation between the sides and angles of a triangle and solution of triangles

102. *To prove that in any triangle the sides are proportional to the sines of the opposite angles,*

$$\textit{viz.,}\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.\quad \textbf{(Sine formula)}$$

Let ABC be a triangle. (In Fig. 1 it is acute-angled and in Fig. 2 obtuse-angled.)

Draw AD perpendicular to BC or BC produced.

In the triangle ABD, AD=c sin B.

In the triangle ACD, AD=b sin C, (Fig. 1)

$AD = b \sin (180° - C)$ (Fig. 2) $= b \sin C$.

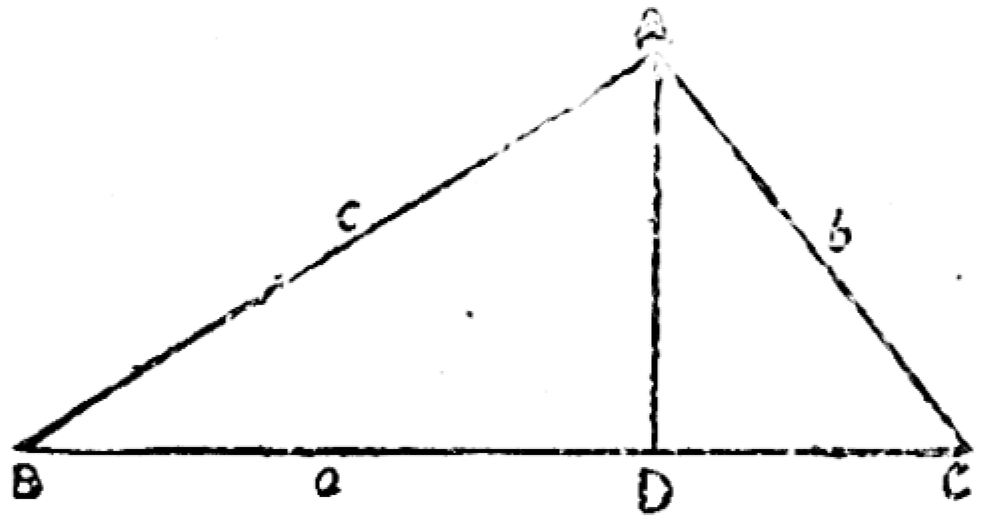


Fig. 1.

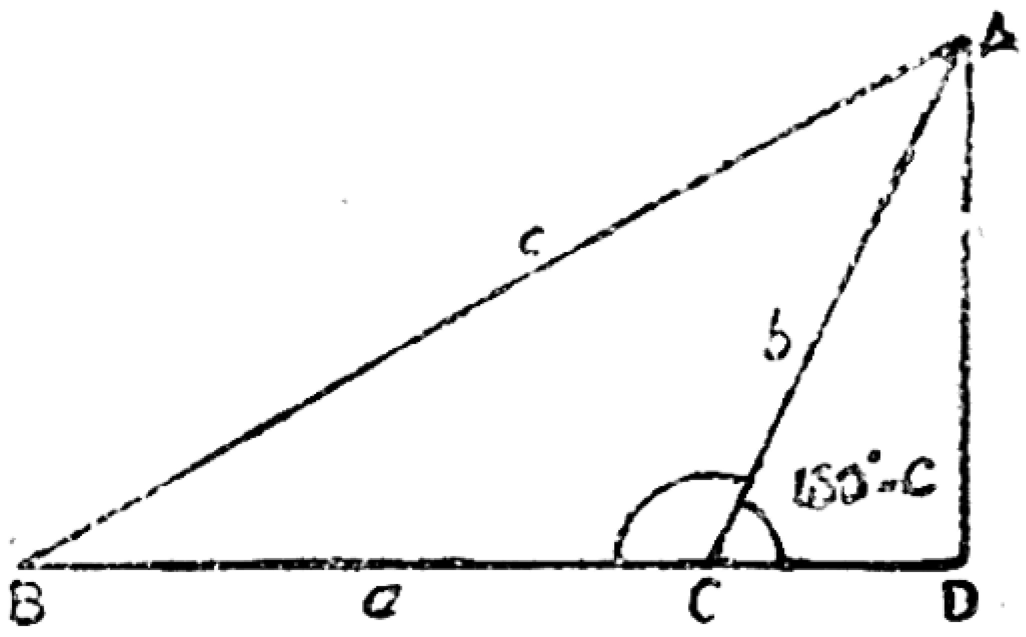


Fig. 2.

Thus $c \sin B = b \sin C$.

$$\therefore \quad \frac{c}{\sin C} = \frac{b}{\sin B}$$

Similarly by drawing perpendicular from C on AB it can be proved that $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$.

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

103. *Given two sides and the included angle of a triangle, to express the third side in terms of these.*

In the figures of the preceding article, suppose b, a and C are given.

We are required to express c in terms of b, a, and C.

In Fig. 1, $\quad CD = b \cos C$,

$$BD = BC - CD = a - b \cos C.$$

$$AD = b \sin C$$

In Fig. 2, $\quad CD = b \cos (180° - C) = -b \cos C$,

$$\therefore \quad BD = BC + CD = a - b \cos C.$$

$$AD = b \sin (180° - C) = b \sin C$$

Thus in both the figures

$$BD = a - b \cos C$$

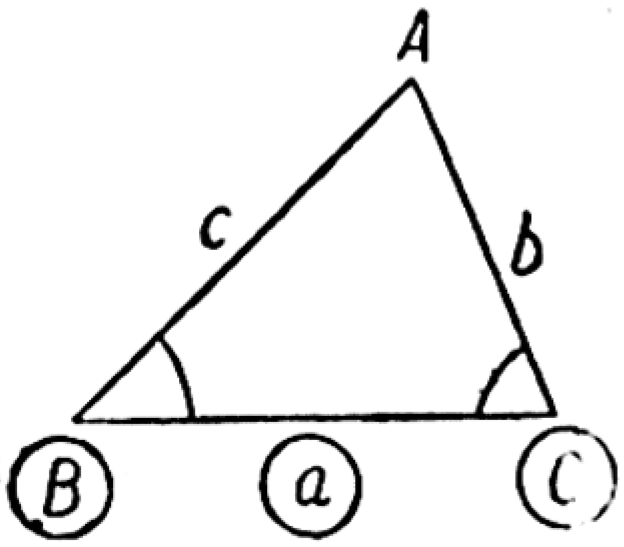
and $$\mathrm{AD}=b\sin\,\mathrm{C},$$

$$c^2=\mathrm{BD}^2+\mathrm{AD}^2=(a-b\cos\mathrm{C})^2+(b\sin\mathrm{C})^2$$
$$=a^2-2ab\cos\mathrm{C}+b^2\cos^2\mathrm{D}+b^2\sin^2\mathrm{C}$$
$$=a^2+b^2(\cos^2\mathrm{C}-\sin^2\mathrm{C})-2ab\cos\mathrm{C}$$
$$\therefore\quad c^2=a^2+b^2-2ab\cos\mathrm{C}\quad \textbf{[Cosine Formula]}$$

Thus the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other sides and cosine of the opposite angle.

103 (*a*)*. **Solution of Oblique-angled triangles.** When no angle of a triangle is a right angle, the triangle is called an '*Oblique-angled triangle*'. When any three of the six parts of such a triangle, one at least being a side, are known, the remaining three parts can be found and thus the triangle can be solved completely. Several different cases arise for solution, but we shall take up only two simple cases here, the others being beyond the scope of this book.

Case I. To solve a triangle when one side and two angles are given.



Since in any triangle $A+B+C=2$ rt. $\angle$s., the third angle can be calculated immediately when two of them are known.

When the angles are all known, the other two sides can be found with the help of the sine rule (one side being already known),

$$\frac{a}{\sin\mathrm{A}}=\frac{b}{\sin\mathrm{B}}=\frac{c}{\sin\mathrm{C}}.$$

using tables, if necessary.

Example 1. Given $\mathrm{B}=30^\circ$, $\mathrm{C}=45^\circ$, $a=6$, solve the triangle.

The third angle A is given by

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=180^\circ$$

or $$\mathrm{A}=180^\circ-\mathrm{B}-\mathrm{C}=180^\circ-30^\circ-45^\circ=105^\circ$$

Then from the sine rule : $$\frac{6}{\sin 105^\circ}=\frac{b}{\sin 30^\circ}=\frac{c}{\sin 45^\circ}.$$

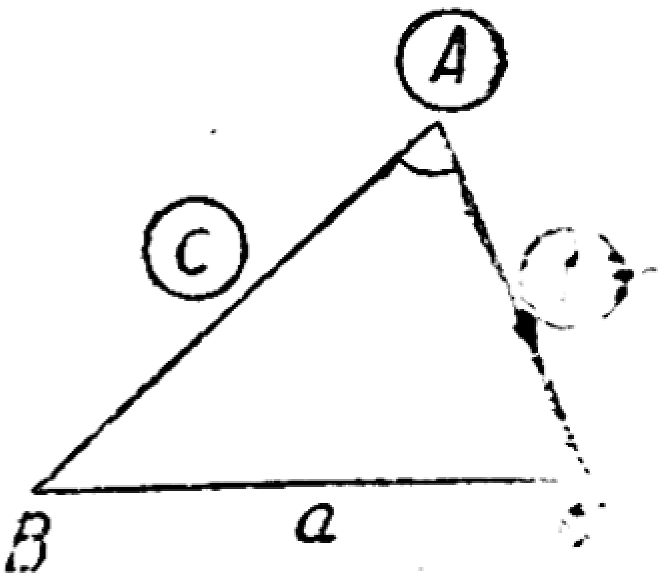
$$\text{Therefore } b=6\ \frac{\sin 30^\circ}{\sin 105^\circ}=6\ \frac{\frac{1}{2}}{\sin\ (180^\circ-75^\circ)}=\frac{3}{\sin 75^\circ}$$

$$=\frac{3}{\cdot 9659}\ \text{(using tables)}=3\cdot 1\ \text{approx.}$$

$$\text{Also}\qquad c=6\times\frac{\sin 45^\circ}{\sin 105^\circ}=6\times\frac{\sin 45^\circ}{\sin 75^\circ}$$

$$=6\times\frac{\cdot 7071}{\cdot 9659}\ \text{using tables}=4\cdot 4\ \text{approx.}$$

Case II. To solve a triangle when two sides and the included angle are given.



To be definite, let the sides b and c and the included angle A be given. The third side a can be calculated immediately from the cosine formula : $a^2=b^2+c^2-2bc$ cos A.

Three sides and one angle are now known, the other two angles can be calculated from the sine formula

$$\frac{a}{\sin \mathrm{A}}=\frac{b}{\sin \mathrm{B}}=\frac{c}{\sin \mathrm{C}}$$

using tables, if necessary.

Example 2. Given $b=3$, $c=5$, $\mathrm{A}=60^\circ$, to solve the triangle.

The side a is given by

$$a^2=b^2+c^2-2bc\cos \mathrm{A}=3^2+5^2-2\times 3\times 5\times \cos 60^\circ$$
$$=34-30\times\tfrac{1}{2}=34-15=19.$$

Thus $a=\sqrt{19}=4\cdot 36$ approx.

The angle B is given by $\dfrac{b}{\sin \mathrm{B}}=\dfrac{a}{\sin \mathrm{A}}$

$$\text{or}\qquad \sin \mathrm{B}=\frac{b}{a}\ \sin \mathrm{A}=\frac{3}{4\cdot 36}\ \sin 60^\circ=\frac{3\times\cdot 866}{4\cdot 36}=\cdot 5958$$

$$\text{or}\qquad \mathrm{B}=36^\circ\ 35'\ \text{approx. (using tables).}$$

The third angle C can now be found from

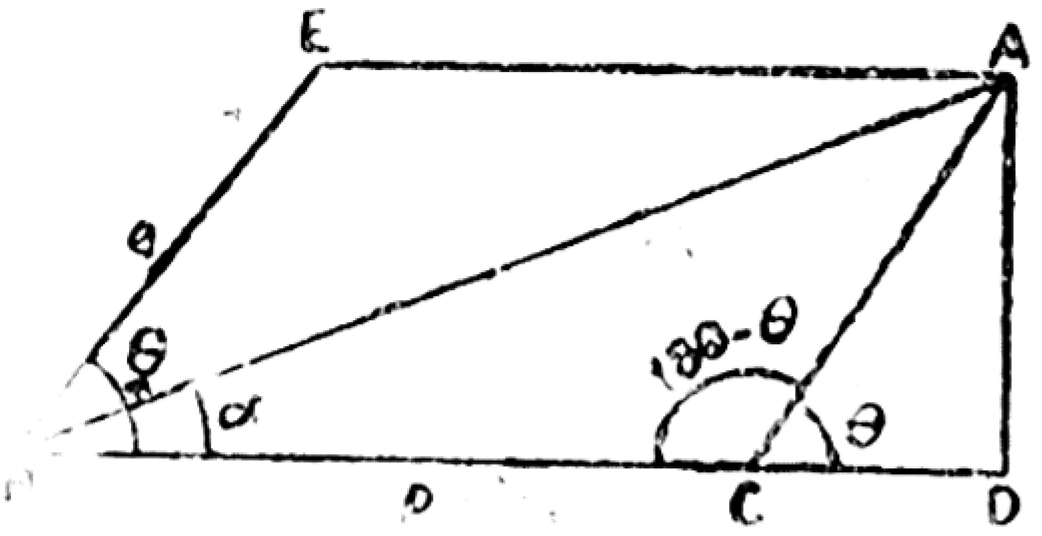
$$C = 180° - A - B = 180° - 60° - 36°\ 35' = 83°\ 25'.$$

LAW OF PARALLELOGRAM OF FORCES

104. If two forces acting simultaneously on a particle are represented in direction and magnitude by the adjacent sides of a parallelogram, the resultant is represented in direction and magnitude by the diagonal of the parallelogram passing through the point of intersection of the sides.

(*For Proof of this law see some book on Mechanics.*)

To find the resultant of two forces P and Q inclined at an angle θ to each other.



Let BC and BE represent the two forces P and Q respectively. Complete the parallelogram BCAE. Suppose ∠EBC $=\theta$.

$$CA = BE = Q.$$

Draw AD prependicular from A to BC or BC produced.

∠ACD is also equal to θ.

$$\therefore \angle BCA = 180° - \theta.$$

In the triangle ABC the sides BC, AC and ∠BCA are given,

$$\therefore \quad AB^2 = P^2 + Q^2 - 2PQ\cos(180° - \theta)$$

$$= P^2 + Q^2 + 2PQ\cos\theta.$$

If α is the inclination of AB with BC,

$$AD = AC\sin\theta = Q\sin\theta$$

$$\sin\alpha = \frac{AD}{AB} = \frac{Q\sin\theta}{AB}.$$

Q and θ are given, AB has been found, therefore the value of α can be calculated.

Or if AB is not found first,

$$\tan\alpha = \frac{AD}{BD} = \frac{Q\sin\theta}{BC + CD} = \frac{Q\sin\theta}{P + Q\cos\theta}.$$

P, Q and θ are known, therefore α can be found.

Example 3. Find the magnitude and direction of the resultant of two forces equal to 3 and 7 units, inclined at an angle of 81°47′ to each other.

In the figure of the last article if R is the length of the diagonal

$$\begin{aligned} R^2 &= P^2+Q^2+2PQ\cos\theta \\ &= 9+49+2\times3\times7\times\sin 8^\circ13' \quad [\cos 81^\circ47'=\sin 8^\circ13'.] \\ &= 58+42\times{\cdot}1429=64{\cdot}002. \end{aligned}$$

$$\therefore\quad R=8{\cdot}000.$$

If α is the angle made by the diagonal with the side 3,

$$\sin\alpha=\frac{7\sin 81^\circ47'}{8}=\frac{7\times{\cdot}9887}{8}={\cdot}866,\ \therefore\ \alpha=60^\circ.$$

Example 4. Given that the sides of a parallelogram are 10 and 5 and the angle between them is equal to 134°. Find the length and inclination of the diagonal passing through the given angle.

$$\begin{aligned} R^2 &= P^2+Q^2 2+PQ\cos\theta \\ &= 100+25+2\times10\times5\times\cos 134^\circ \\ &= 125-100\sin 44^\circ \end{aligned}$$

$$[\because\ \cos 134^\circ=-\cos(180^\circ-134^\circ)=-\cos 46^\circ=-\sin 44$$

$$=125-100\times{\cdot}6947=125-69{\cdot}47=55{\cdot}53,$$

$$\therefore\ R=\sqrt{55{\cdot}53}$$

$$\log R=\tfrac{1}{2}\log 55{\cdot}53=\tfrac{1}{2}(1{\cdot}7445)={\cdot}8723.$$

$$\therefore\ R=7{\cdot}452.$$

If α is the inclination with the side 10,

$$\tan\alpha=\frac{5\sin 134^\circ}{10+5\cos 134^\circ}=\frac{5\sin 46^\circ}{10-5\sin 44^\circ}$$

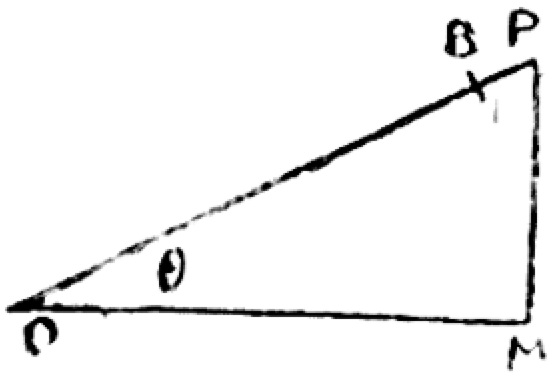
$$=\frac{5\times{\cdot}7193}{10-5\times{\cdot}6947}=\frac{{\cdot}7193}{1{\cdot}3053}.$$

$$\log\tan\alpha=\bar{1}{\cdot}8569-{\cdot}1156=\bar{1}{\cdot}7413.$$

$$\tan\alpha={\cdot}5512,\ \therefore\ \alpha=28^\circ52'.$$

VALUES OF TRIGONOMETRICAL RATIOS FOR VERY SMALL ANGLES

105. Values of cos θ when θ is very small.



Let POM be a right-angled triangle with angle POM$=\theta$. From OP cut off OB$=$OM.

$$\cos\theta=\frac{OM}{OP}=\frac{OB}{OP}=\frac{OP-BP}{OP}=1-\frac{BP}{OP}.$$

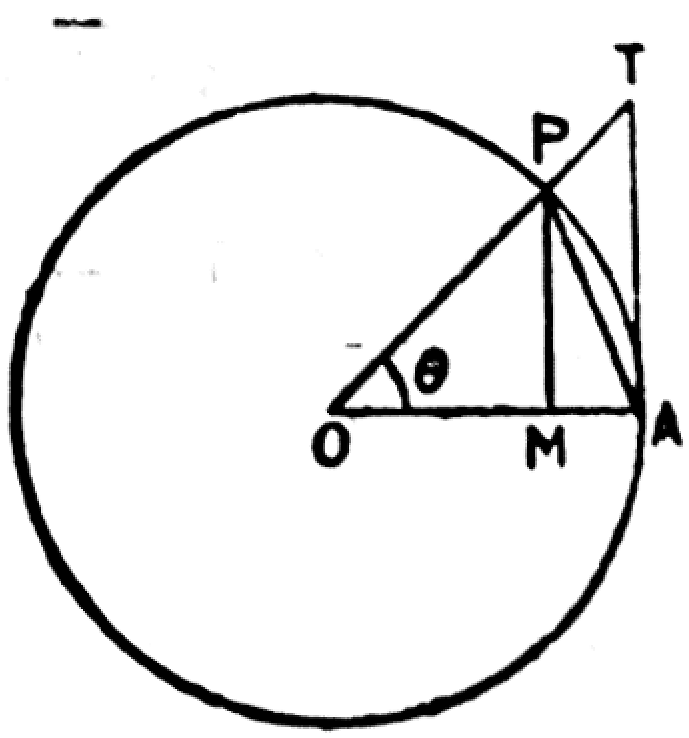
As θ diminishes, the difference between OP and OM, *viz.*, BP becomes smaller and smaller and when θ is sufficiently small, $\frac{BP}{OP}$ becomes negligible in comparison with 1, therefore $\cos\theta=1$ approx. for small values of θ.

[Note. $\cos\theta=1$ is only an approximation and not a mathematically correct result.

On consulting the tables, we find that where a result correct up to three decimal places is required this approximation is correct for all values of θ less than 1°·8, and if a result correct only to two places of decimals is needed the approximation may be used up to 5°·5 and so on. Thus for different degrees of accuracy there are different limits to the value of θ up to which θ may be considered so small that $\cos\theta=1$.

106. Values for sin θ and tan θ for very small angles.

Draw a circle of radius r with O as centre and take an angle AOP. Let θ be its *radian measure*. Draw PM perpendicular to OA and AT tangent to the circle at A. Produce OP to meet AT at T and draw the chord PA.



$$PM=OP\sin\theta \quad =r\sin\theta$$

$$AT=OA\tan\theta \quad =r\tan\theta$$

Area of $\triangle$ POA$=\frac{1}{2}$ OA$\times$PM $=\frac{1}{2}r^2\sin\theta$

Area of $\triangle$ TOA$=\frac{1}{2}$ OA$\times$AT $=\frac{1}{2}r^2\tan\theta$.

If θ is nth part of 2π (360°), the area of the sector POA is $\frac{1}{n}$ times the area of the circle.

$$\theta = \frac{2\pi}{n}, \quad \therefore \; n = \frac{2\pi}{\theta}.$$

$\therefore$ area of the sector $POA = \frac{\pi r^2}{n} = \frac{\pi r^2 \theta}{2\pi} = \frac{1}{2} r^2 \theta.$

Area of the $\triangle$ POA is evidently less than that of the sector POA which is again less than that of the $\triangle$ TOA.

i.e., $\therefore \; \frac{1}{2} r^2 \sin\theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan\theta$

$$\sin\theta < \theta < \tan\theta \qquad \ldots \qquad \ldots \qquad (i)$$

Dividing all these by $\sin\theta$

$$1 < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta}.$$

$$\therefore \; 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

i.e., $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$.

But when θ is very small $\cos\theta = 1$.

$\therefore \; \frac{\theta}{\sin\theta}$ lies between 1 and 1, *i.e.*, it is equal to 1,

i.e., $\sin\theta = \theta$ for very small values of θ.

Dividing all the terms of (i) by $\tan\theta$ we have

$$\sin\theta \times \frac{\cos\theta}{\sin\theta} < \frac{\theta}{\tan\theta} < 1,$$

$$\therefore \; \cos\theta < \frac{\theta}{\tan\theta} < 1 \cdot$$

i.e., $\frac{\theta}{\tan\theta} = 1$ for small values of θ or $\tan\theta = \theta$.

i.e., **$\sin\theta = \tan\theta = \theta$ when θ is very small.**

Caution : θ is the *radian measure* of the angle and **not the sexagesimal measure.**

Radian measure of $5° = \frac{5 \times \pi}{180} = \cdot 0873$ and that of $10° = \cdot\mathbf{1746}$.

$\therefore$ θ for $5° = \cdot 0873$ and θ for $10° = \cdot 1746$,

$\sin 5° = \cdot 0872$ and $\sin 10° = \cdot 1736$.

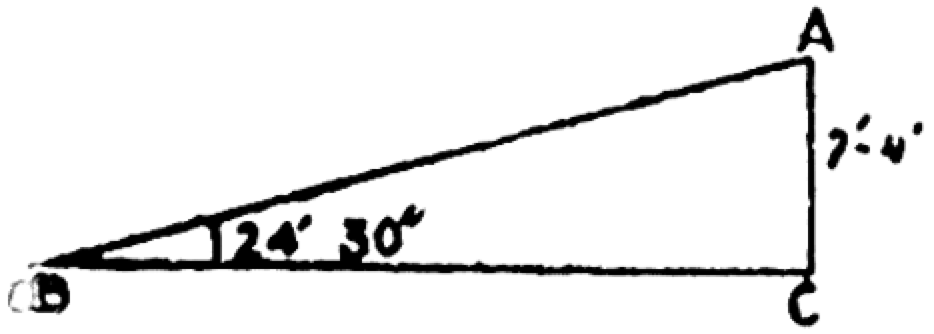
Up to three places of decimals $\sin\theta=\theta$ is true for 5° and even for slightly greater angles. Up to two places of decimals this approximation is correct up to 10°.

$$\tan 5^\circ = \cdot 0875 \text{ and } \tan 10^\circ = \cdot 1763.$$

For an accuracy up to three decimal places $\tan\theta=\theta$ is just true for 5° and up to two places of decimals 10° becomes a little too large.

Generally speaking angles up to 5° may be taken to be sufficiently small for ordinary degree of accuracy in these approximations.

Example 5. From the top of a wall 7 ft. 4 inches high the angle of depression of an object on the ground is 24′30″, find its distance from the wall.



Let AC be the wall and B the object.

$\angle ABC = 24'\ 30''$.

$$\tan 24'\ 30'' = \frac{AC}{BC}, \quad \therefore\ BC = \frac{AC}{\tan 24'\ 30''} = \frac{AC}{\theta}, \text{ where}$$

θ is the radian measure of 24′ 30″, which is so very small.

$$\theta = \frac{49}{120} \times \frac{3\cdot142}{180} \quad \text{and} \quad AC = \frac{22}{3} \text{ ft.}$$

$$\therefore \quad BC = \frac{\frac{22}{3}}{\frac{49\times 3\cdot142}{120\times 180}} = \frac{22\times120\times180}{3\times49\times3\cdot142} \text{ ft.}$$

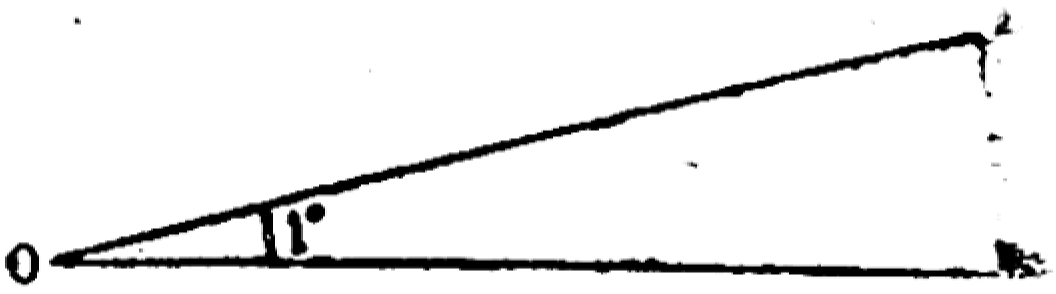
$$\log\ BC = (1\cdot3424 + 2\cdot0792 + 2\cdot2553 - \cdot4771 - 1\cdot6902 - \cdot4972)$$
$$= 3\cdot0124.$$

$$\therefore \quad BC = 1029 \text{ ft.} = 343 \text{ yds.}$$

107. Angle subtended by an object at a given point is the angle between the straight lines joining the extremities of the object to the given point. Objects are seen by rays of light proceeding in straight lines, and passing through the narrow hole in the eye known as the pupil which may be considered to be just a point for all practical purposes.

Example 6. Find the length of a rod which subtends an angle equal to 1° at the eye of an observer at a distance of 100 ft.

Let O be the observer and AB the rod held perpendicular to the line of sight. With O as centre and OA as radius draw an arc of a circle, $\angle AOB$ is very small therefore AB is very small compared with the distance OA, hence the point B will lie on this circle and the chord AC will practically coincide with the arc AB.



If θ is the radian measure of $\angle AOB$, $AB = OA \times \theta$,

$$\therefore \quad AB = 100 \times \frac{1 \times 3{\cdot}142}{180} \text{ feet} = 1{\cdot}745 \text{ ft.}$$

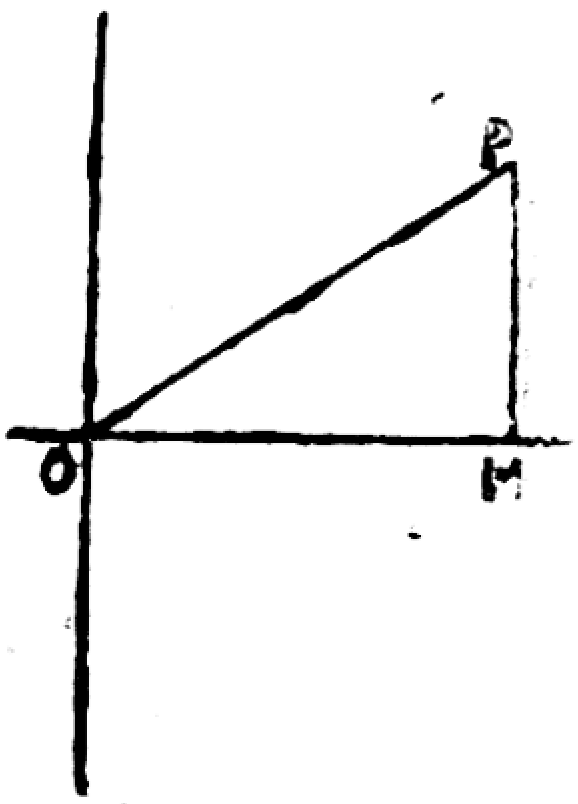
Example 7. Find sine 30°·1

$$\begin{aligned} \sin 30^\circ{\cdot}1 &= \sin (30^\circ + {\cdot}1^\circ) \\ &= \sin 30^\circ \cos {\cdot}1^\circ + \cos 30^\circ \sin {\cdot}1^\circ \\ &= \tfrac{1}{2} \times 1 + \frac{\sqrt{3}}{2} \times \frac{{\cdot}1 \times \pi}{180} = {\cdot}5 + {\cdot}866 \times {\cdot}0017 \\ &= {\cdot}5 + {\cdot}0015 = {\cdot}5015. \end{aligned}$$

108. Limits of sine, cosine and tangent.

(*i*) $\quad \text{sine} = \dfrac{\text{ordinate}}{\text{radius vector}}.$

The length of the ordinate MP is always less than the radius vector OP (the hypotenuse of the right-angled $\triangle$ POM) and it may have any positive or negative value numerically *less* than OP. Sine of an angle may thus have any positive value *numerically* less than unity and it lies therefore between +1 and −1.



(*ii*) $\text{cosine} = \dfrac{\text{abscissa}}{\text{radius vector}}.$

Abscissa OM may have any positive or negative value less than the radius vector OP, therefore the cosine also lies between $+1$ and -1.

(*iii*) tangent$-\dfrac{\text{ordinate}}{\text{abscissa}}$.

The ordinate PM may be less or greater than the abscissa OM and may have any value however small or however large in comparison. One of these may be positive and the other negative, therefore the tangent of an angle may have any value positive or negative between $+\infty$ and $-\infty$.

The graphs of sin, cos and tan will also illustrate these limits in a beautiful manner.

Example 8. Can an angle θ exist such that $2\sin^2\theta+\sin\theta$ shall be equal to 6 ?

If

$$2\sin^2\theta+\sin+\theta=6$$

$$2\sin^2\theta+\sin\theta-6=0$$

or

$$\sin\theta=\frac{-1\pm\sqrt{1+48}}{4}=\frac{-1\pm7}{4}=\tfrac{3}{2}\text{ or }-2$$

$\sin\theta$ cannot have a value numerically greater than 1, therefore no value of θ can exist which may make this relation possible.

GRAPHS OF SINE, COSINE AND TANGENT

109. *To draw the graph of* **sin** θ.

1st Method—We knew the values of $\sin\theta$ for certain values of θ *viz.* 0°, 30°, 45° etc. in the first quadrant. Values of sin θ for some values of θ in the second quadrant can be calculated from the relation, $\sin(180°-\theta)=\sin\theta$.

$$\sin 120°=\sin(180°-60°)=\sin 60°=\frac{\sqrt{3}}{2}=\cdot 87,$$

$$\sin 135°=\sin(180°-45°)=\sin 45°=\frac{1}{\sqrt{2}}=\cdot 71,$$

$$\sin 150°=\sin(180°-30°)=\sin 30°=\tfrac{1}{2}=\cdot 50,$$

$$\sin 180°=\sin(180°-0°)=\sin 0°=0.$$

Values of $\sin\theta$ for certain values of θ in the 3rd and 4th quadrants are found as under—

$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\cdot 50.$

$\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\cdot 71.$

$\sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\cdot 87.$

$\sin 270^\circ = \sin(180^\circ + 90^\circ) = -\sin 90^\circ = -1.$

$\sin 300^\circ = \sin(180^\circ + 120^\circ) = -\sin 120^\circ = -\cdot 87.$

$\sin 315^\circ = \sin(180^\circ + 135^\circ) = -\sin 135^\circ = -\cdot 71.$

$\sin 330^\circ = \sin(180^\circ + 150^\circ) = -\sin 150^\circ = -\cdot 50.$

$\sin 360^\circ = \sin 0^\circ \ldots\ldots\ldots\ldots\ldots\ldots = 0$

Tabulating all these values we have—

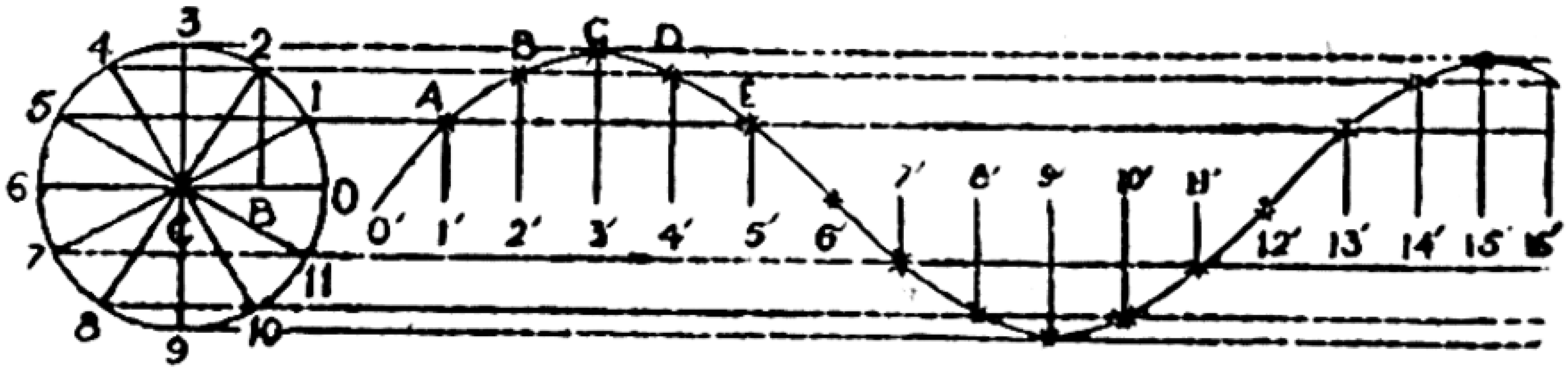
θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$	0	·50	·71	·87	1	·87	·71	·50	0

θ	210°	225°	240°	270°	300°	315°	330	360°
$\sin\theta$	—·50	—·71	—·87	—1	—87	—·71	—·50	0

Draw the axes of x and y on a squared paper and choose a convenient scale along the axis of x to represent θ and also a scale along the axis of y to represent $\sin\theta$ (a scale in which unity on the axis of y is equivalent to about 60° along the axis of x, gives a good shape of the curve.) Plot these values and join the points by a smooth curve.

2nd Method (Geometrical).

Draw a circle of unit radius (any convenient length may be taken to be unity) preferably with its centre on the line which is to represent the axis of x, and divide the circumference into an equal number of parts (say 12). Mark them as 0, 1, 2, etc., beginning from the right hand extremity of the horizontal diameter and mark equidistant points, 0′, 1′, 2′,



etc. along the axis of x. At these points draw ordinates equal to the corresponding heights of the points, 0, 1, 2, etc. of the circle above the axis of x.

[Lines parallel to the axis of x have been drawn from the points of division of the circle to facilitate the drawing of the corresponding ordinates. On a graph paper they are not necessary.]

Join the extremities of these ordinates by a smooth curve. This curve represents the graph of sin θ.

Proof. If C is the centre of the circle, angles 1C0, 2C1, 3C2 are all equal to $\frac{360°}{12}=30°$. The points 0′, 1′, 2′ etc. on the axis of x are also equidistant.

From any point of division of the circle (say 2) draw 2B perpendicular to the horizontal diameter.

$$\sin 2C0=\frac{B2}{C2}=B2 \text{ because C2, the radius, is unity.}$$

Along the axis of x the distance 0′2′ represents the angle 2C0 and the ordinate of 2′ is equal to 2B, therefore the ordinate of 2′ represents sin 2C0. Similarly the ordinates of the points, 0′, 1′, 2′, etc., on the axis of x represent the sines of the corresponding angles 0, 1C0, 2C0, etc. in the circle, therefore the abscissae of the curve represent angles and the ordinates, the sines of these angles. Thus it is the graph of sin θ.

The point 12′ on the axis corresponds to 0 on the circle, and one complete revolution (360°) is represented by the line 0′—12′. Angles greater than 360° are represented by the abscissae 0′—13′, 0′—14′ etc. and these angles are represented in the circle by 1C0, 2C0 etc. again in the second revolution.

It is clear that the graph is a continuous curve. The height gradually rises to its maximum value, +1 (at 3′) and then after passing through a value 0 (at 6′) reaches its maximum value, −1 (at 9′), and completes the cycle at 12′. Continued further the curve is endlessly repeated.

The curve extended right and left resembles in appearance the ripples produced on the surface of still water by a stone, or the waves in sea. One wave means the complete cycle from 0′ to 12′ and this distance is called **a wave length**. The distances between two identical points like 1′, 13′ : 2′, 14′ etc., are also equal to one wave length.

Such curves are also called **harmonic curves**.

110. *To draw the graph of cos θ.*

1st Method.

cos (180°−θ) and cos (180°+θ)=−cos θ, therefore from the known values of cosines of 0°, 30°, 45°, 60° and 90° we can calculate the cosines of 120°, 135°, 150° 180°, 210° etc.

Tabulating these results we have—

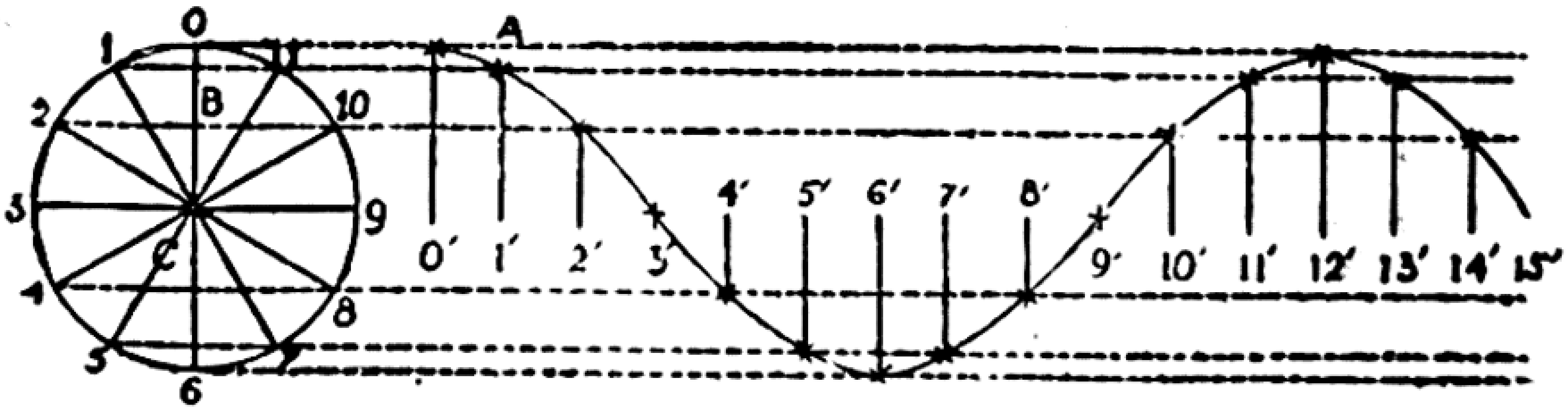
θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
cos θ	1	·87	·71	·50	0	−·50	−·71	−·87	−1

θ	21°	225°	240°	270°	300°	315°	330°	360°
cos θ	−·87	−·71	−·50	0	·50	·71	·87	1

Plot these points with θ along the axis of x and cos θ along the axis of y and join the points by a smooth curve.

2nd Method (Geometrical).

Draw a circle as in the case of sines and divide it in the same manner. Mark these points 0, 1, 2, etc., beginning from the upper extremity of the vertical diameter. Mark equidistant points 0′, 1′, 2′, 3′, etc., along the axis of x and at these points draw-ordinates corresponding to the heights of 0, 1, 2 etc. on the circle. The extremities of these ordinates joined by a



smooth curve give the graph of cos θ.

Proof.—Take any point of division on the circle and draw a perpendicular from this point to the vertical diameter (as 2B).

$$\cos 2C0 = \frac{BC}{2C} = BC \text{ because the radius 2C is unity.}$$

The ardinate of the point 2′ is also equal to BC by construction. Similarly the ordinates of the points 0′, 1′, 2′ etc., represent the cosines of ∠s 0, 1C0, 2C0, etc. Therefore the curve represents the graph of cos θ. As in the case of sines the curve has been extended to points 13′, 14′, 15′, etc., to represent angles grater than 360°.

111. In principle of construction the graph of cos θ is similar to that of sin θ and differs only in taking the upper end of the vertical diameter as the starting point instead of the right hand end of the horizontal diameter. The forms of the curves are also similar except that 0′ of the cosine curve corresponds to the point 3′ of the sine curve. This cosine curve is obtained by shifting the sine curve through 90° to the left. This is also a harmonic curve.

The wave length as before is equal to 0′−12′; 1′−13′; 2′−14′, etc.

112. *To draw the graph of tan θ.*

1st Method. ($\tan 180° - \theta$) $= -\tan\theta$ and $\tan(180° + \theta) = \tan\theta$, therefore the tangents of 120°, 135°, 150°, etc., can be calculated from the known values of tangents of 30°, 45°, 60°, etc.

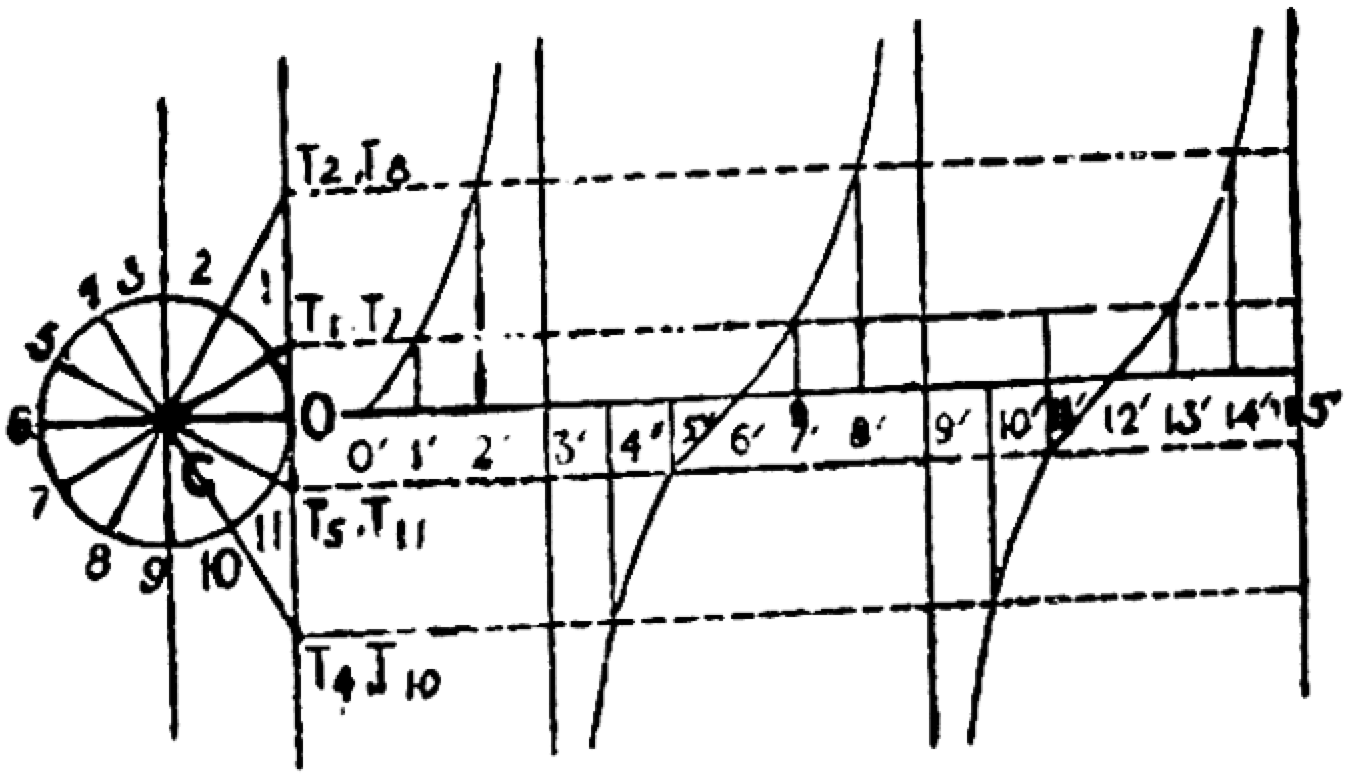
The following table gives these values :—

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
tan θ	0	·58	1	1·73	∞	−1·73	−1	−·58	0

θ	210°	225°	240°	270°	300°	315°	330°	360
tan θ	·58	1	1·73	∞	−1·73	−1	−·58	0

We can plot these values on a graph paper and draw the graph.

2nd Method (*Geometrical*). Draw a circle of unit radius as before. Divide the circumference and mark the points of division as 0, 1, 2, etc., beginning from the right hand extremity of the horizontal diameter.



Draw a tangent to the circle at the point 0 and produce the radii C1, C2, C3, etc., forwards or backwards as necessary to meet this tangent at T_1; T_2, etc.

Mark 0′, 1′, 2′, 3′ etc , equidistant points on the axis of x and draw ordinates at these points equal to 0, $0T_1$, $0T_2$, etc. respectively.

Radii C3 and C9 are parallel to the tangent, therefore they meet it at $+\infty$ and $-\infty$ respectively.

Join the extremities of the ordinates by smooth curves. These curves represent the required graph.

Proof. Take any points of division (say 2) and let the radius C2 meet the tangent at T_2.

$$\tan 2C0=\frac{0T_2}{0C}=0T_2.$$

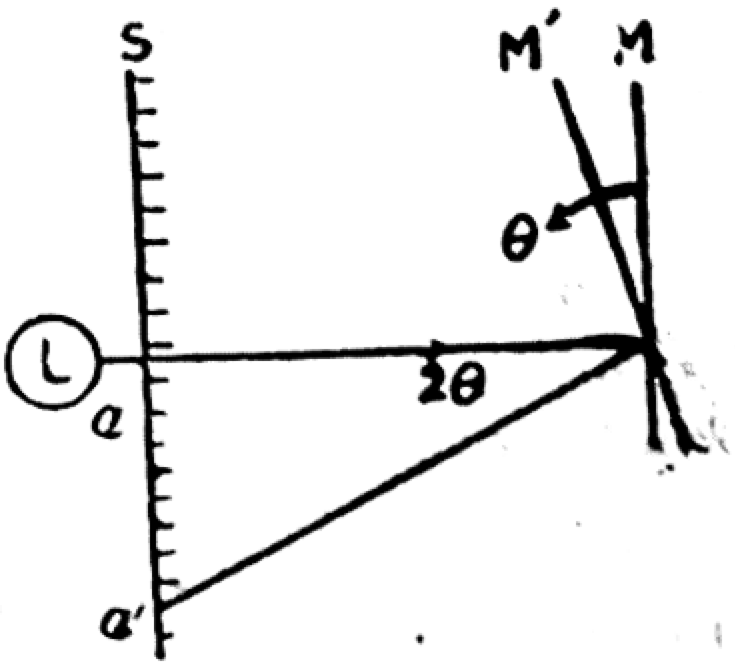
The ordinate of point 2′ is also equal to $0T_2$.

Similarly the ordinates of the points 0′, 1′, 2′, etc., represent the tangents of angles, 0, 1C0, 2C0 represented by the abscissae 0′, 0′ −1′, −2′ etc. Hence the curve is the graph of tan θ.

The graph of tan θ consists of a series of branches extending from $-\infty$ to $+\infty$ and repeating themselves at intervals of 180°.

PRACTICAL APPLICATIONS

113. Lamp and scale arrangement. Suppose a magnet, a coil of wire or any other body suspended by a fibre or held by a spring etc., is rotated by some kind of force. The simplest method of measuring the angle of rotation is to attach a pointer to the body and to fix a suitable circular scale under the pointer. The movement of the pointer over the scale will indicate the deflection. This arrangement is found in a sensitive balance, a static galvanometer, voltmeter, etc. Sometimes the suspended system is too delicate to bear the weight of the pointer or the deflection is too small to be read on an ordinary scale, therefore an arrangement known as *lamp and scale* method is used. It consists of a small mirror M attached to a suitable point on the body and a scale S with a slit *a* to allow a narrow beam of light from a lamp L to fall on the mirror. The light reflected by the mirror is taken on the scale S. If the mirror M in its undisturbed state is perpendicular to the path of light, the reflected beam will retrace its path of light, and will meet the scale at *a*. When the suspended body rotates, the mirror being rigidly attached to it rotates along with it through the same angle (θ) and the reflected beam will rotate through twice



this angle (2θ) (see some book on light). Let M′ be the new position of the mirror and suppose that the reflected beam now meets the scale at a'.

$$\tan 2\theta = \frac{a'a}{\text{distance from } a \text{ to the mirror}}$$

$a'a$ is read on the scale and the distance between the slit and the mirror is measured by a metre rod. $\tan 2\theta$ having thus been determined the value of 2θ is found from the table of tangents. Half of this gives the required value of deflection.

When angles are very small $\tan 2\theta = 2\theta$ radians, which can be easily converted into degrees and the use of tables can be avoided but if tables are available, it is more convenient in practice to find the value of 2θ from them.

The lamp L is sometimes replaced by a telescope through which the reflection of a' is seen, or if the distance between the scale and the mirror is not very large, one can see it even by an unaided eye.

Note. A beam of light serves as a long weightless pointer in this lamp and scale arrangement.

Example 9. Find in degrees the deflection of a galvanometer needle, when the spot of light moves through 20 mm. on a scale placed at a distance of one metre from it.

In the figure of the last article

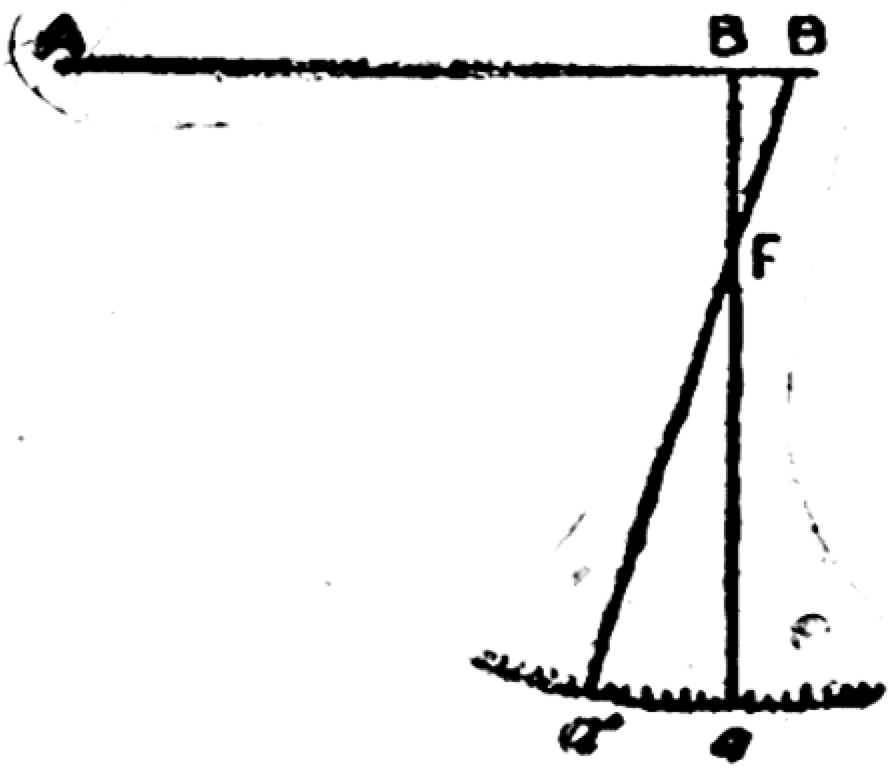
$$\tan 2\theta = \tfrac{20}{1000} = \cdot 02,$$

$$\therefore \quad 2\theta = \cdot 02 \text{ radians}$$

$$= \frac{\cdot 02 \times 180^\circ}{\pi} = 1^\circ 9'$$

or $$\theta = 34'\cdot 5.$$

114. Magnification by levers. Movements through small distances and angles, hardly perceptible to the naked eye, can be rendered clearly visible and capable of measurement by the help of bars and levers etc. Lamp and scale method of the last article is one example of the type where a beam of light serves as a bar.



Levers. Suppose a metallic rod AB is heated and its length increases to AB′. The increase in length BB′ is generally too small to be measured accurately by an ordinary scale.

If we attach a lever BFa with fulcrum at F to the end B of the rod and keep the end A fixed, the lever will take up a position B′Fa' when expansion takes place. A circular scale fixed under the end a of the lever measures the distance through which a moves and if the lengths of the arms BF and Fa of the lever are known, BB′ can be calculated.

Let $\angle a\mathrm{F}a'=\theta$, $\angle$BFB′ is also equal to θ. BB′ may be taken to be the arc of a circle with FB as radius because θ is very small.

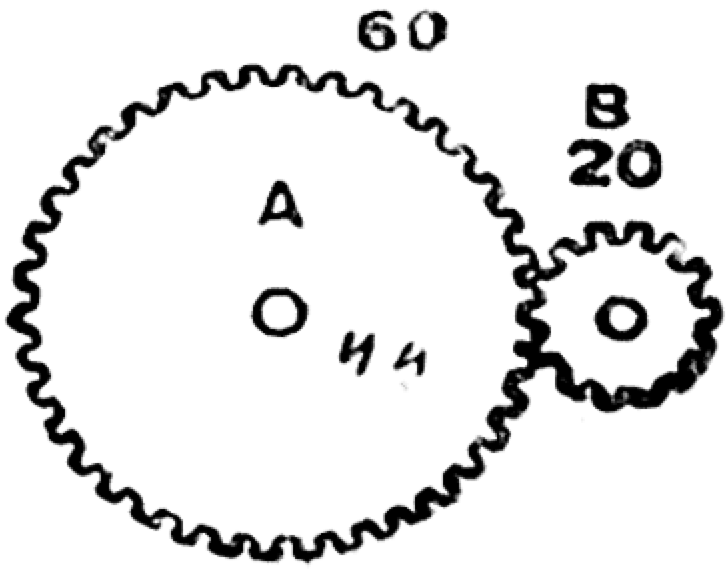
$$\theta=\frac{\mathrm{BB'}}{\mathrm{BF}}=\frac{aa'}{\mathrm{F}a} \qquad \left(\theta=\frac{l}{r}\right)$$

$$\therefore\quad \mathrm{BB'}=aa'\times\frac{\mathrm{BF}}{\mathrm{F}a}.$$

Thus BB′ is determined.

Magnification is equal to $\dfrac{aa'}{\mathrm{BB'}}=\dfrac{\mathrm{F}a}{\mathrm{BF}}$, *viz.*, the ratio between the arms of the lever.

115. Cogged Wheels. Let A and B be two cogged wheels with 60 and 20 cogs respectively having their teeth fitting exactly into each other.



If the wheel A makes one complete revolution, the wheel B will obviously make three revolutions and thus a slow motion of A can be magnified three times. Conversely the fast motion of B can be reduced to one-third in the motion of A. By a chain of such wheels the motion can be magnified or reduced to any extent.

It is by such suitable magnifications and reductions that the seconds, minutes and hours are shown by the different hands of a clock although the motion of all these is controlled by the same pendulum.

Example 10. The arms of a lever are in the ratio 1 : 10. Find the distance through which the end of the longer arm moves when the end of the shorter arm moves through 2 mm.

In the figure of the last article BB′=2 mm.

and $$\frac{Fa}{BF}=10.$$

$$\therefore \quad \frac{aa'}{BB'}=\frac{10}{1},$$

$$\therefore \; aa'=2\times 10 \text{ mm.}=2{\cdot}0 \text{ cm.}$$

Example 11. The number of teeth in two cogged wheels are equal to 90 and 6 respectively. By how much will the smaller wheel revolve if the larger wheel rotates through 30° ?

The ratio of the teeth is 15 : 1, therefore the smaller wheel will rotate through an angle 15 times as much as the larger wheel.

The angle through which the smaller wheel turns is thus $=15\times 30^\circ=450^\circ$ *viz.*, $1\frac{1}{4}$ revolution.

EXAMPLES XVIII

1. In a triangle $a=1$, $b=\sqrt{3}$ and $A=30^\circ$ find B.

$$\left(\textbf{Hint.}\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\right)$$

2. Sides b and c of a triangle are 7 and 13 respectively and $\angle C=60^\circ$. Find the remaining parts.

3. If $B=60^\circ$, $a=3$ and $b=3\sqrt{3}$ find c.

Forces acting at a point and the angles between them are given below. Find the magnitude of the resultant and also the angle which it makes with the first mentioned force.

4. 4, 5 ; 30°.

5. 15, $3\sqrt{2}$; 45°.

6. 7, 9 ; 60°.

7. 11, 14 ; 120°.

8. Find the value of θ from the relation, $\cos^2\theta-5\cos\theta+6=0$.

9. The moon subtends an angle of 31′ 7″ at the eye of an observer on the surface of the earth. At what distance from the eye should a 4 anna bit (dia. 1·95 cm.) be held just to cover the disc of the moon ?

10. The eye cannot distinguish an object if it subtends an angle less than 1′. What is the greatest distance up to which a pice (dia. 1″) can be recognised ?

11. Find the cosine of 60°·5. (*P. U. 1931*)

12. What is a radian ?

Find in seconds the angle subtended by a man 6 ft. tall at a point in the earth one mile away. (*P. U. 1931*)

13. Prove that in any triangle, the sides are proportional to the sines of the opposite angles.

Use this theorem to deduce from the triangle of forces that —"If three forces keep a particle in equilibrium each force is proportional to the sine of the angle between the other two." (*P. U. 1927*)

14. Enunciate the theorem known as the "parallelogram of velocities."

If U and V represent the velocities along the two sides of a parallelogram and θ the angle between them, write down the expression for the direction and magnitude of the resultant. Hence or otherwise determine the actual velocity of a body which possesses simultaneously a velocity $\sqrt{2}$ miles per hour due south and 4 miles per hour due north-east. (*P. U. 1928*)

15. Assuming that $\dfrac{\sin i}{\sin r}$ is constant plot a graph between i and r. Tables should be used for finding the values of r for given values of i.

16. Draw a graph of $\sin\theta$ and θ. (*P. U.*)

17. Draw a graph of $\tan\theta$ and θ. (*P. U.*)

18. One end of a steel rod is fixed and the other presses against an end of a lever 10·5 cm. from the fulcrum. One rod being heated turns the lever through 2°. Find the increase in length of the rod. ($90° = \frac{\pi}{2}$ radians.) (*P. U. 1926*)

19. If two forces equal to 7 and 9 poundals act stimultaneously on a particle in directions inclined at an angle 60° to each other, find the magnitude of the resultant and also the angle it makes with the first force. (*P. U. 1932*)

20. A force of 5 lb. and other force Q act at 60°. They are balanced by a force of 7 lb. Find the magnitude of Q. Prove the formula used. (*P. U. 1936*)

21. Prove that in any triangle $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

Given $\angle B=45°$, $\angle C=75°$, $a=\sqrt{3}$, find b. (*P. U. 1942*)

22. Enunciate the theorem known as the 'parallelogram of velocities.'

If u and v represent the velocities along the sides of a parallelogram and θ the angle between them, obtain expressions for the magnitude and direction of the resultant. (*P. U. 1941*)

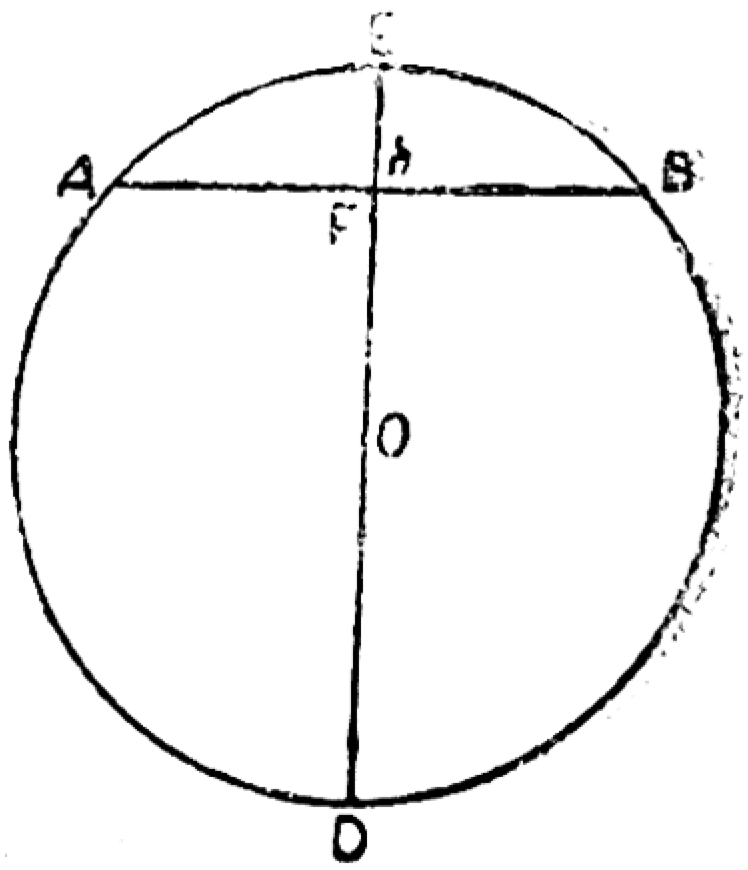
CHAPTER XIX

Spherometer

116. Let a circle AEBD represent a vertical section of a sphere through its centre and suppose a chord AB of this circle represents the horizontal section of the sphere through some point A.

Draw a diameter ED of the circle perpendicular to the chord AB, cutting it at F. Let EF=h and radius OD=R.

AF=FB=r (say) is the radius of the *horizontal section* which is a circle having AB as diameter.



From the geometry of the circle

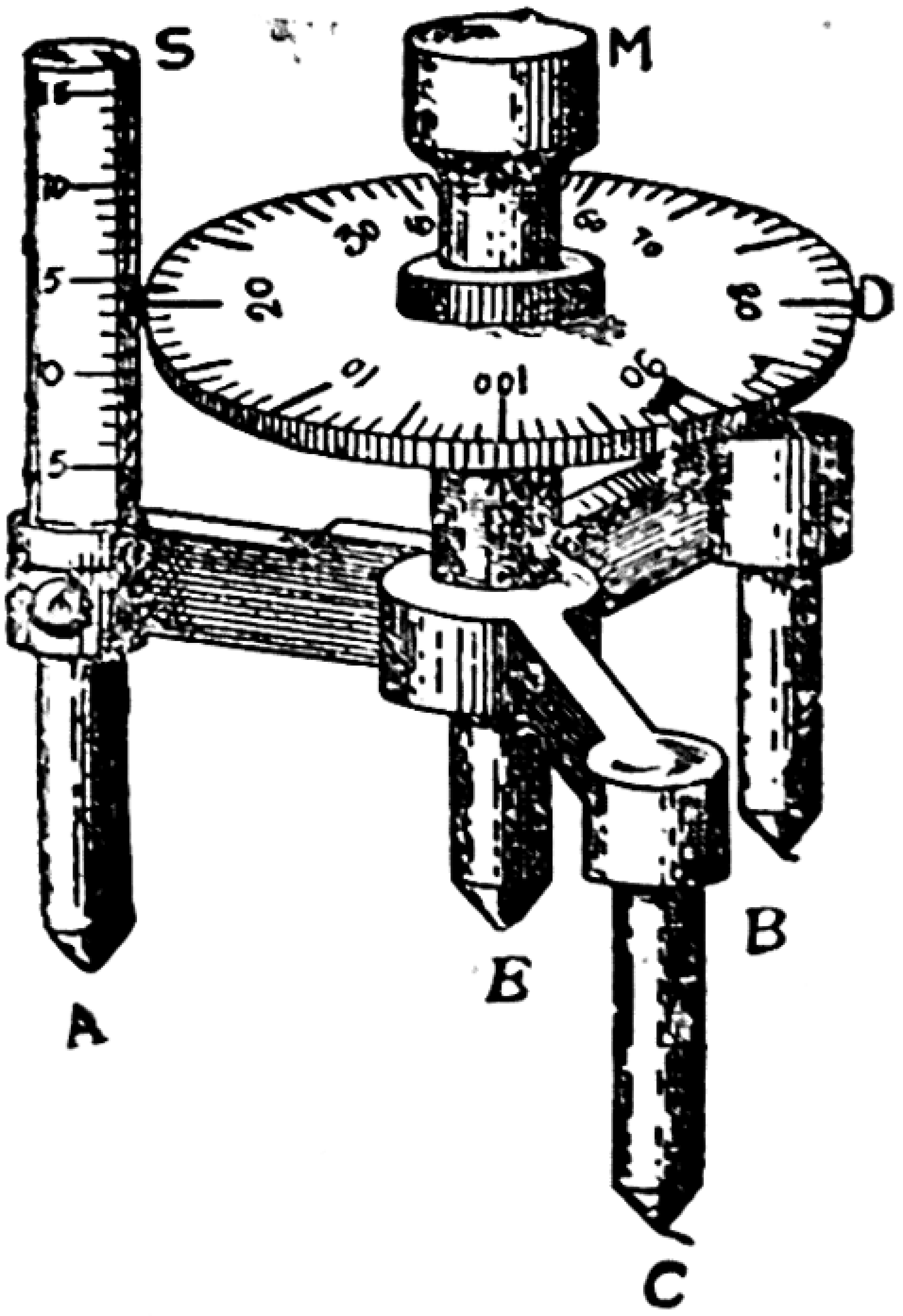
$$AF \times FB = EF \times FD.$$

$$\therefore \quad r^2=h(2R-h)=2Ra-h^2$$

i.e.,
$$R=\frac{r^2}{2h}+\frac{h}{2}$$

Thus if the radius (r) of a section of a sphere and (h) the maximum thickness of the section are known, the radius (R) of the sphere can be calculated.

117. Spherometer. It is an instrument used for measuring small thicknesses and radii of curvature of spherical mirrors and lenses. The spherometer consists of a tripod, the legs of which are of equal length and so adjusted that their feet A, B, C lie at the corners of an equilateral triangle. An accurately turned screw passes through the centre of this tripod perpendicular to the plane of its base and carries a milled head M and a circular disc D graduated into a certain number of equal divisions (generally 100). A vertical scale S divided into millimetres is attached to one arm of the tripod with its divisions close to the edge of the disc.

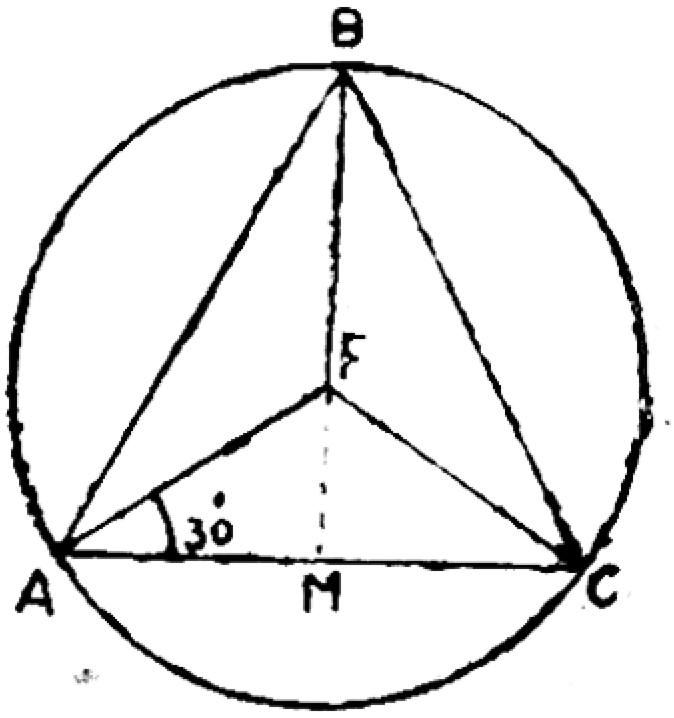


The pitch of the screw is such that one complete revolution of the disc moves the point E of the screw by half a millimetre or one millimetre. If the pitch is ·5 millimetre and the disc is divided into 100 parts the rotation of the disc by one division means the movement of the point of the screw forwards or backwards by $\frac{\cdot 5}{100} = 005$ millimetre.

Measurement of Radius of Curvature.

At the first the sphrometer is placed on a *plane* glass surface and the screw is rotated by the milled head till its point just touches the surface when the instrument begins to spin round the central screw. The scale and the disc are read and this reading is called the zero reading, *viz.*, the reading when the feet of the legs and the point of the screw are in the same plane.

After moving up the screw to allow for the convexity of the surface (if convex), the instrumen is placed on the surface whose radius of curvature is to be measured. The screw is again moved down till its point just touches the spherical surface as indicatad again by the spinning of the instrument. The difference between this reading and the previous reading gives the distance by which the topmost point of the curved surface is higher than the plane of the feet of the instrument, *viz.*, the value of EF or h of the previous article.



Let a circle ABC represent the section of the sphere passing through the feet A, B, C of the spherometer and suppose F is the centre of this circle.

ABC is an equilateral triangle of side l (say). Join FA, FC, and FB. Draw FM perpendicular to AC.

$$FA=FB=FC=r\ ;\ \angle FAM=30^\circ$$

and $$AM=\frac{AC}{2}=\frac{l}{2}$$

$$\cos 30^\circ=\frac{AM}{AF},\ \therefore\ AF=\frac{AM}{\cos 30^\circ}=\frac{\frac{l}{2}}{\frac{\sqrt{3}}{2}}=\frac{l}{\sqrt{3}}.$$

Thus the value of r of the last article is known in terms of the distance between the feet.

$$\therefore \qquad R=\frac{\left(\frac{l}{\sqrt{3}}\right)^2}{2h}+\frac{h}{2}=\frac{l^2}{6h}+\frac{h}{2}.$$

h is measured by the spherometer as explained above and l is measured by a metre rod and thus the value of R can be calculated.

If the surface is concave the spherometer measures the depth FE′ as h, where E′ is the foot the screw now touching the concave surface, and the formula is the same.

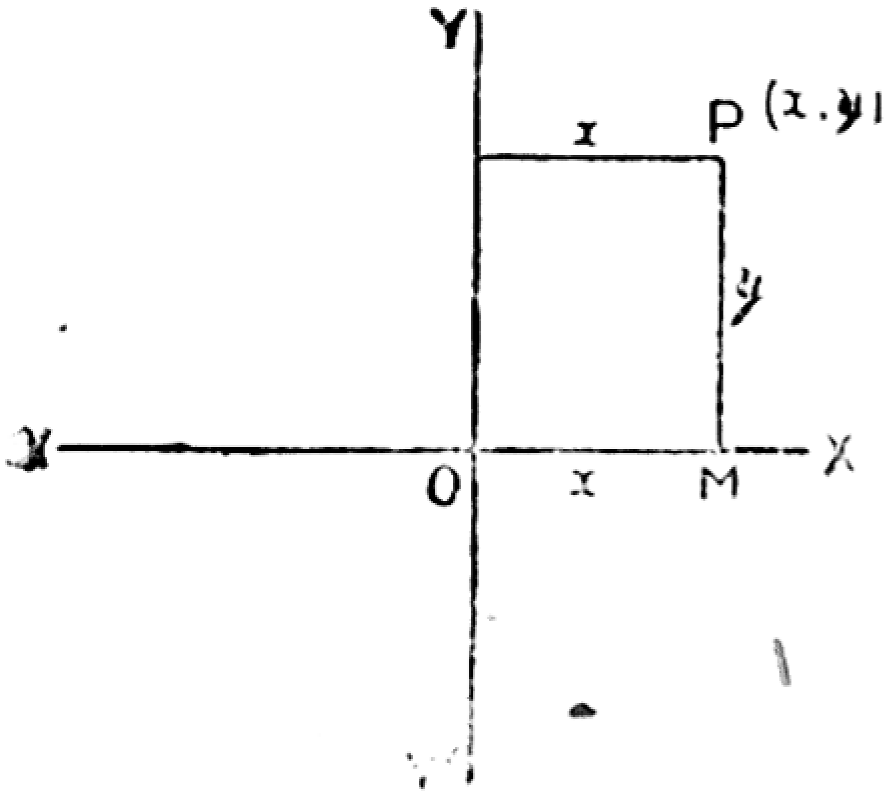
CHAPTER XX

Co-ordinates

118. The position of a point can be fixed in a plane in many ways, Two of them known as Cartesian Co-ordinates and Polar Co-ordinates are described below.

CARTESIAN CO-ORDINATES

119. Let XOX′ and YOY′ be two lines at right angles to each other intersecting at O and suppose P is a point in the same plane at a distance y from the line XOX′ and x from the line YOY′.

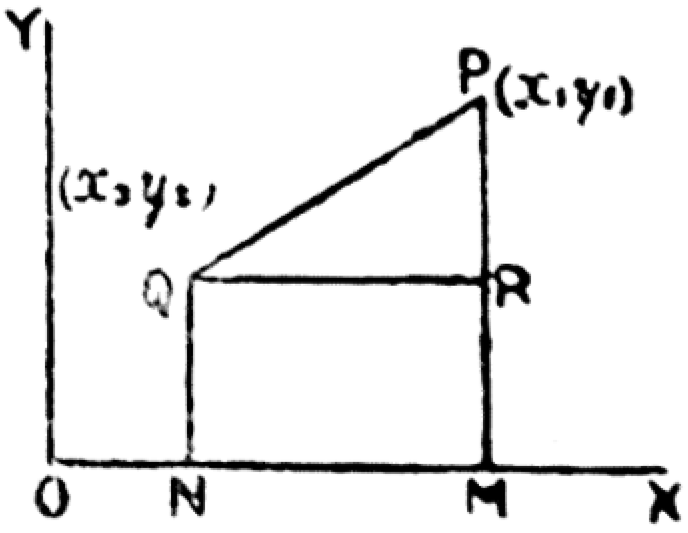


As explained before, the lines XOX′ and YOY′ are called the **axes** and the point O is known as the **origin**. The distances x and y *viz*, **abscissa** and **ordinate** are called the **Cartesian Co-ordinates** of the point P.

Unknown co-ordinates are generally denoted by the letters x and y and the known ones by x', y', x_1, y_1, etc.

The points are denoted by the values denoting their co-ordinates. The point **(x, y)** is the point whose abscissa is x and ordinate is y (the point P of the figure).

120. *To express the distance between two points in terms of their cartesian co-ordinates.*



Let P(x_1, y_1) and Q (x_2, y_2) be the given points.

Draw PM and QN the ordinates of P and Q and QR perpedicular to PM.

OM$=x_1$ and ON$=x_2$.

PM$=y_1$ and QN$=y_2$

QR=NM=OM−ON$=x_1-x_2$.

PR=PM−RM=PM−QN$=y_1-y_2$.

$$PQ^2=QR^2+PR^2=(x_1-x_2)^2+(y_1-y_2)^2.$$

$$\therefore \qquad PQ=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

Example 1. Find the distance between the points (7, 6) and (3, −2).

$$\text{The distance}=\sqrt{\{7-3\}^2+\{6-(-2)\}^2}$$
$$=\sqrt{4^2+8^2}$$
$$=\sqrt{80}=8{\cdot}944.$$

Example 2. Show that the points (2, −2), (8, 4), (5, 7) and (−1, 1) lie at the corners of a rectangle.

Distance between 1st and 2nd point

$$=\sqrt{(2-8)^2+(-2-4)^2}=\sqrt{72}.$$

Distance between 2nd and 3rd point

$$=\sqrt{(8-5)^2+(4-7)^2}=\sqrt{18}.$$

Distance between 3rd and 4th point

$$=\sqrt{(5+1)^2+(7-1)^2}=\sqrt{72}.$$

Distance between 4th and 1st point

$$=\sqrt{(-1-2)^2+(1+2)^2}=\sqrt{18}.$$

This shows that the quadrilateral formed by joining those points in order has its alternate sides equal *viz.*, it is a parallelogram.

Distance between 1st and 3rd point

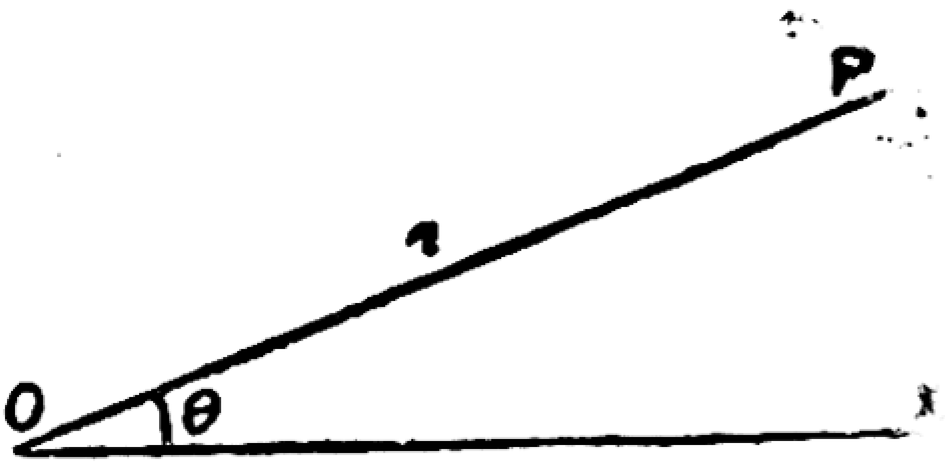
$$=\sqrt{(2-5)^2+(-2-7)^2}=\sqrt{90}.$$

Distance between 2nd and 4th point

$$=\sqrt{(8+1)^2+(4-1)^2}=\sqrt{90}.$$

Diagonals are equal, therefore it is a rectangle.

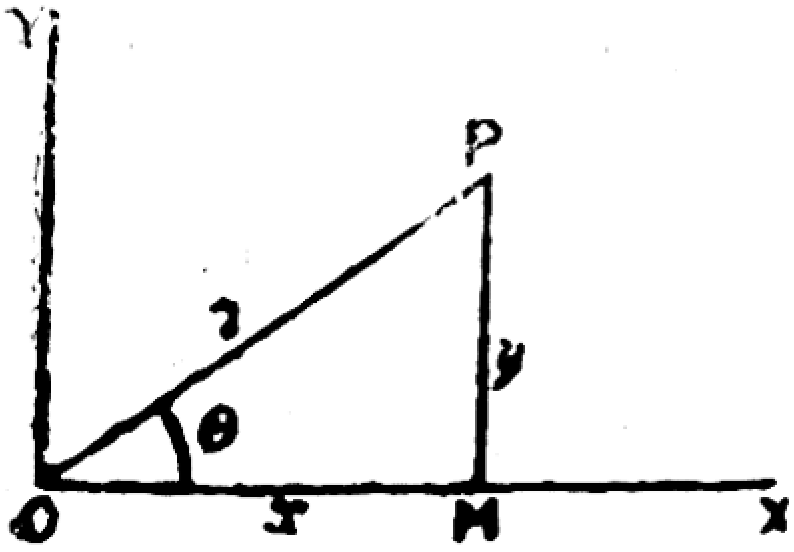
POLAR CO-ORDINATES



121. Let O be a fixed point and OX a fixed straight line in the given plane.

The position of any point P in the plane is completely determined if distance OP (r) and $\angle$XOP (θ) are known. r and θ are called the **polar co-ordinates** of P, with respect to the point O known as the pole and line OX, the initial line. Length r is called **radius vector** and θ the **vectorial angle.**

122. Relations between the cartesian co-ordinates and the polar co-ordinates of a point.



Let P be a point having x and y as cartesian co-ordinates with respect to OX and OY and r and θ as polar co-ordinates with respect to O and OX.

(*i*) Cartesian co-ordinates in terms of polar co-ordinates :—

$$x=r\ \cos\theta,\ y=r\sin\theta.$$

(*ii*) Polar co-ordinates in terms of cartesian co-ordinates :—

$$r=\sqrt{x^2+y^2} \text{ and } \tan\theta=\frac{y}{x} \textit{ viz. } \theta=\tan^{-1}\frac{y}{x}.$$

The formula proved in the previous article and all other formulae in terms of cartesian co-ordinates can be expressed in terms of polar co-ordinates either directly or by the help of the relations given above.

123. *To find the distance between two points in terms of their polar coordinates.*

Let P and Q be the two points with r_1, θ_1 and r_2, θ_2 as their polar co-ordinates. Join PO, QO and PQ.

$\angle POX=\theta_1$ and $\angle QOX=\theta_2$,

$\therefore\ \angle QOP=\theta_2-\theta_1$.

In the $\triangle$ QOP

$PQ=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)$, [Art 103.

$\therefore\ PQ=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}$.

EXAMPLES XX

Find the distances between the following pairs of points correct up to two places of decimals.

1. (2, 1) and (—2, —1).

2. (2, 3) and (4, —5).

3. (6, 8) and (2, 4).

4. (4, 3) and origin.

5. (—4, —2) and (—6, 3).

6. (5, 2) and (—9, —3).

7. Find the value of x_1 if the distance between $(x_1, 2)$ and (3, 4)=8.

8. Prove that the points (71, 71), (27, 9), (0, 0) and (—13, —1) lie on a circle with centre (—13, 84) and radius 85.

[**Hint** : Show that the distances of the given points from (—13, 84) are=85 each.]

9. Show that the points (2, 4), (2, 6) and $(2+\sqrt{3}, 5)$ form an equilateral triangle.

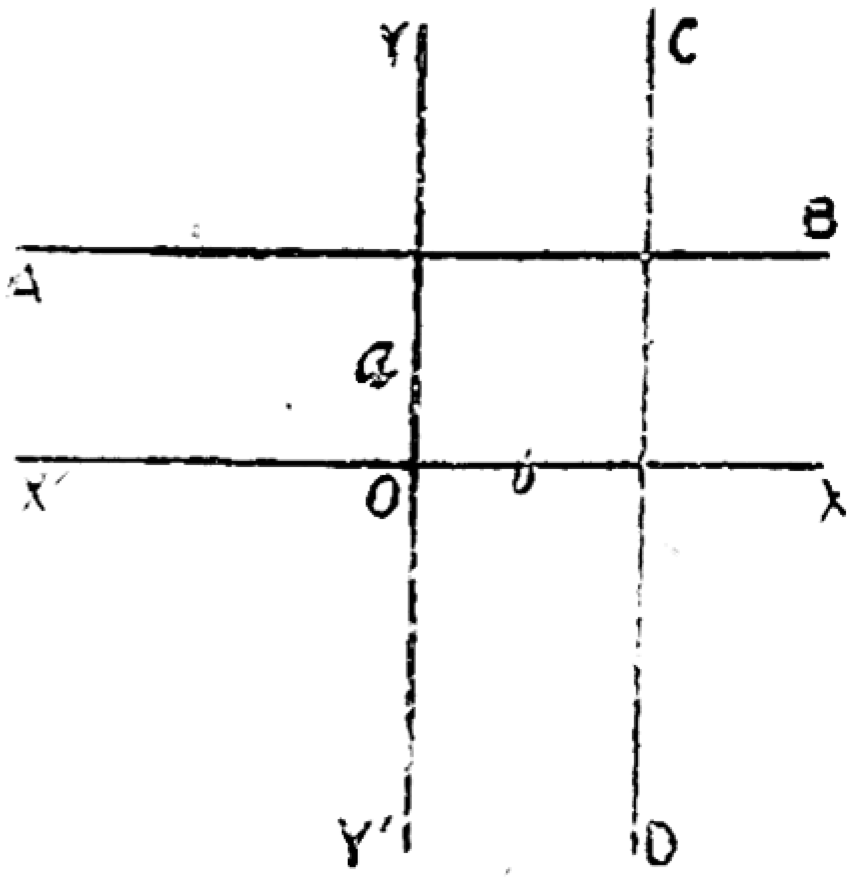
10. Prove that the points (2, 3), (6, 3), (6, 7), (2, 7) are the corners of a square whose side is equal to 4.

11. Show that the points (9, 12), $(\sqrt{29}, 14)$, and $(10, 5\sqrt{5})$ are equidistant from the origin.

CHAPTER XXI

The Straight Line

124. *To find the equation to a straight line parallel to the axis of x.*



Let AB be a straight line parallel to the axis of x.

The distance of any point on AB from XX′ *viz.*, the ordinate is constant say equal to a.

$\therefore$ $y=a$ is the required equation.

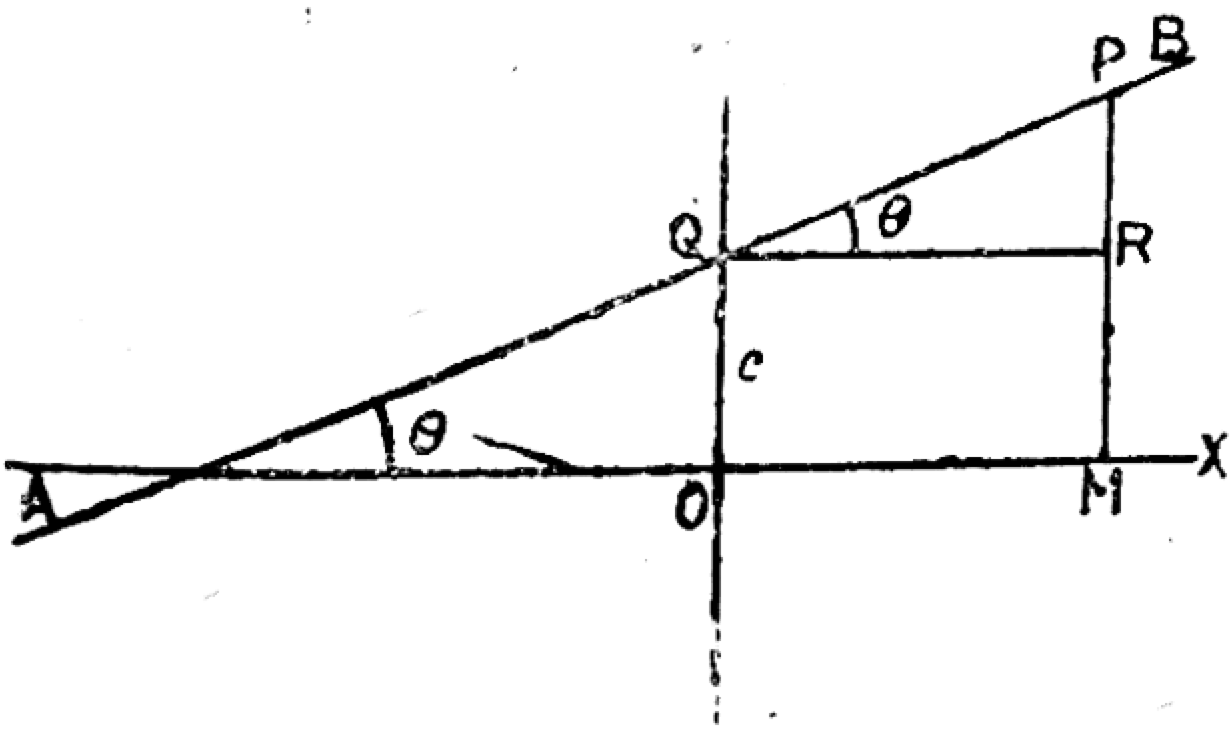
125. *To find the equation to a straight line parallel to the axis of y.*

Let CD be such a line. The distance of any point on it from YY′ *viz.*, the abscissa is constant, say equal to b.

$\therefore$ $x=b$ is the required equation.

126. *To find the equation to a straight line in terms of its intercept on the axis of y and its inclination to the axis of x.*

Let AB be a straight line cutting off an intercept OQ$=c$ on the axis of y and having an inclination θ to the axis of x.



Take any point P (x, y) on this line and draw PM its ordinate. Draw QR parallel to OX meeting PM at R.

$\angle$PQR is also equal to θ.

$$\frac{\text{PR}}{\text{QR}}=\tan\theta\ ;$$

$$\therefore \text{PR}=\text{PQ}\tan\theta=x\tan\theta$$

$$y=\text{PM}=\text{PR}+\text{RM}$$

$$=x\tan\theta+\text{OQ}$$

$$=x\tan\theta+c.$$

$\therefore$ $y+x\tan\theta+c$ is the required equation.

If $\tan\theta$ is put as m the equation becomes

$$y=mx+c.$$

x and y are interchangeable and an equation $x=my+c$ will also represent a straight line with c an intercept on the axis of x and m the tangent of the angle made with the axis of y.

Cor. If the line passes through the origin, $c=0$.

$\therefore$ $y=mx$ is the equation to a straight line passing through the origin.

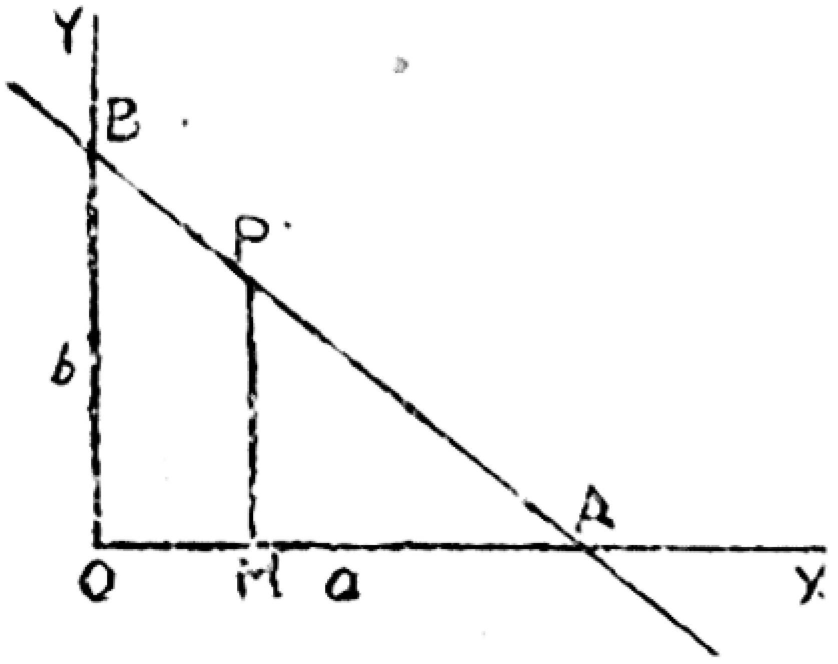
The equation is called the **tangent form** or **slope form** of the equation to a straight line.

Example 1. Find the equation to a straight line making an angle of 45° with the axis of x and cutting the axis of y at a distance of 4 below the origin.

In this case $m=\tan 45^\circ=1$ and $c=-4$.

$\therefore$ $y=x-4$ is the required equation.

127. *To find the equation to a straight line in terms of its intercepts on the axes.*



Let AB be a straight line cutting the axes at A and B.

Suppose OA$=a$ and OB$=b$.

If P (x, y) is any point on this line, draw the ordinate PM.

$\triangle$s ABO and APM are similar.

$$\therefore \frac{\text{PM}}{\text{BO}}=\frac{\text{MA}}{\text{OA}}=\frac{\text{OA}-\text{OM}}{\text{OA}}=1-\frac{\text{OM}}{\text{OA}}$$

$$i.e., \frac{y}{b}=1-\frac{x}{a},$$

$$\therefore \quad \frac{x}{a}+\frac{y}{b}=1 \text{ is the required equation.}$$

This equation is known as the **intercept form** of the equation.

Example 2. Find the equation to the straight line which cuts off intercepts equal to -5 and 6 on the axes of x and y respectively.

Let $\frac{x}{b}+\frac{y}{b}=1$ be the required equation.

The values a and b are -5 and 6 respectively.

$$\therefore \quad \frac{x}{-5}+\frac{y}{6}=1,$$

$\therefore$ $6x-5y+30=0$ is the required equation.

128. Two forms of the equation to a straight line have been found above : *viz.*

$$y=mx+c \text{ and } \frac{x}{a}+\frac{y}{b}=1.$$

It will be noticed that both these forms are equations of the first degree. It can be shown that any other equation of the first degree represents a straight line. **Ax+By+C=0,** where A, B and C are some constants, is the general equation of the first degree.

129. Different forms of the equation to a straight line can be easily transformed from one form to another.

(*a*) *To express the tangent form in the intercept form and vice versa.*

(*i*) $y=mx+c$

$$\therefore \qquad \frac{y}{c}=\frac{mx}{c}+1, \text{ or } \frac{-m}{c}+\frac{y}{c}=1$$

$$\text{or} \qquad \frac{x}{-\frac{c}{m}}+\frac{y}{c}=1 \text{ (the intercept form).}$$

(*ii*) $\frac{x}{a}+\frac{y}{b}=1.$

$$\therefore \quad \frac{bx}{a}+y=b, \text{ or } y=\left(-\frac{b}{a}\right)x+b \text{ (the tangent form).}$$

(*b*) *To express $Ax+By+C=0$ in the intercept and tangent forms.*

(*i*) $\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0$

$$\therefore \quad \frac{\mathrm{A}}{\mathrm{C}}x+\frac{\mathrm{B}}{\mathrm{C}}y=-$$

$$\therefore \quad \frac{x}{-\frac{\mathrm{C}}{\mathrm{A}}}+\frac{y}{-\frac{\mathrm{C}}{\mathrm{B}}}=1 \text{ (the intercept form).}$$

(*ii*) $\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0$

$$\therefore \qquad \mathrm{B}y=-\mathrm{A}x-\mathrm{C}$$

$$y=\left(-\frac{\mathrm{A}}{\mathrm{B}}\right)x-\frac{\mathrm{C}}{\mathrm{B}} \text{ (the tangent form).}$$

The latter form of the equation represents a straight line having an intercept $-\frac{C}{B}$ on the axis of y and inclined to the axis of x at an angle θ such that $\tan\theta=-\frac{A}{B}$, therefore $Ax+By+C=0$ also represents a straight line.

130. An equation to a straight line passing through two given points can be found as under.

Find the equation to a straight line passing through (2, 3) and (−4, 1).

Let $y=mx+c$ be the equation,

Points (2, 3) and (−4, 1) lie on the line.

$$\therefore \quad 3=2m+c \ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$1=-4m+c \ldots\ldots\ldots\ldots\ldots (ii)$$

Subtracting (*ii*) from (*i*)

$$6m=2 \text{ or } m=\tfrac{1}{3}$$

Substituting this value of m in (*i*) we get

$$3=\tfrac{2}{3}+c, \quad \therefore \quad c=\tfrac{7}{3}.$$

m and c are thus determined,

$$\therefore \quad y=\tfrac{1}{3}x+\tfrac{7}{3}.$$

or $\quad 3y-x-7=0$ is the required equation.

Instead of supposing the equation to be $y=mx+c$, we can suppose it to be $\frac{x}{a}+\frac{y}{b}=1$ and find the value of the constants a and b. Final equation will be the same in both cases.

***131. Collinear Points.** Three points may be proved to be collinear by finding the equation to the straight line through two of them and showing that the co-ordinates of the third point also satisfy it.

Example 3. Show that the points (1, 2), (7, 8) and (5, 6) lie on a straight line.

Let $y=mx+c$ be the equation to the straight line passing through the 1st and 2nd points.

$$2=m\times 1+c \ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$8=m\times 7+c \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$\therefore \quad m=1 \text{ and } c=1,$$

and the equation is $y=x+1$.

The co-ordinates 5 and 6 of the third point also satisfy this equation, therefore the third point also lies on the straight line joining the first two.

132. Points of Intersection of lines The point of intersection of two lines lies on both the lines, therefore the co-ordinates of this point must satisfy the equations of both of them. The finding of the point of intersection of two lines thus means the finding of the values of x and y which will satisfy both the eqnations, *i. e*, the solution of the given equations.

Example 4. Find the point of intersection of the lines.

$$4x+3y=10 \ldots\ldots\ldots\ldots (i)$$

$$3x+5y=13 \ldots\ldots\ldots\ldots (ii)$$

Multiplying (i) by 3 and (ii) by 4 and subtract

$$11y+22. \quad \therefore \quad y=2.$$

Substituting this in (i) : $4x+6=10$, $\therefore$ $x=1$.

Thus (1, 2) is the point of intersection.

133. Concurrency of lines. If three or more lines are concurrent, *viz.*, pass through one point, we should be able to find a point the co-ordinates of which satisfy the equations of all of them. In practice the point of intersection of two of the lines is determined and if the co-ordinates of this point satisfy the equations of the other lines also, all those lines pass through the same point.

Example 5. Show that the following lines are concurrent—

$$2x-5y+1=0 \ldots\ldots\ldots\ldots\ldots (i)$$

$$x+3y+6=0 \ldots\ldots\ldots\ldots\ldots (ii)$$

$$x+y+4=0 \ldots\ldots\ldots\ldots\ldots (iii)$$

Take equations (ii) and (iii).

Subtracting (iii) from (ii) we get : $2y+2=0$ or $y=-1$.

Substituting this in (iii) : $x+3=0$ or $x=-3$.

By trial we find that equation (i) is also satisfied by these values of x and y.

Hence the point $(-3,-1)$ iies on all the three lines, *viz.*, the lines are concurrent.

***134.** An equation of the first degree represents a straight line. Equations of the second degree generally represent some curves, but some of them represent a pair of straight lines.

Consider the equation $x^2-3xy-4y^2=0$.

The equation may be written as $(x-4y)(x+y)=0$.

This equation is satisfied if $x-4y=0$ or $x+y=0$.

The quadratic equation is thus reduced to two equations of the first degree, therefore it represents two straight lines given by these equations.

As a rule, if a quadratic equation can be expressed as a product of two factors of the first degree, the equation will represent two straight lines.

If the factors are not apparent as above, we can use the formula $\left(x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2}\right)$ for finding the linear equations.

Example 6. Find the straight lines represented by

$$y^2-6xy+5x^2=0$$

$$y=\frac{6x\pm\sqrt{36x^2-20x^2}}{2}=\frac{6x\pm4x}{2}=5x \text{ or } x,$$

$\therefore$ $y=5x$ and $y=x$ are the required straight lines.

EXAMPLES XXI

Find the equations to the straight lines whose intercepts on the axis of y and inclinations to the axis of x are :—

1. 1 and 45°. **2.** -2 and 30°.

Find the inclinations to the axis of x of the straight lines represented by the following equations :—

3. $3x+3y+2=0$. **4.** $5x+3y+15=0$.

Find the equations to the straight lines passing through the following pairs of points :—

5. (0, 0) and (2, −2). **6.** (−1, 3) and (6, −7).

7. (−3, 0) and (0, −5).

Find the equations to the straight lines whose intercepts on the axes of x and y respectively are :—

8. 2 and 3. **9.** 1 and $\frac{2}{3}$. **10.** −4 and −4.

11. Find the equations to the sides of a triangle, the coordinates of whose angular points are (0, 1), (2, 0) and (−1, −2).

*__12.__ Prove that the points (1,−9), $(1,-\frac{23}{3})$ and (4, −5) are collinear.

*__13.__ Show that the points (−3, 1), $(-\frac{13}{3}, 2)$ and (1,−2) lie in a straight line.

Express the following equations in the intercept form :—

14. $4x+3y-15=0.$ **15.** $5x+2y+10=0.$ **16.** $4x-3y=2.$

Find the points of intersection of the following pairs of lines :—

17. $8x-5y-1=0$ and $7x-4y+1=0.$

18. $3x+5y+7=0$ and $4x-3y-10=0.$

19. Find the equation to the straight line joining (1,2) and the point of intersection of the lines $3x-y-3=0$ and $4x+2y-14=0.$

20. Find the equation to the straight line passing through the origin and the point of intersection of the lines $2x+y+1=0$ and $x+\frac{y}{3}+1=0.$

*Show that the following sets of lines are concurrent :—

21. $x+y+5=0.$ $2x+y+7=0.$ $x+2y+8=0.$

22. $x-y=0.$ $2x-3y+1=0.$ $3x-2y=1.$

23. $9x-6y=3.$ $x+2y=11.$ $y-x=1.$

Show that each of the following equations represents a pair of straight lines :—

*__24.__ $x^2-4y^2=0.$ **25.** $y^2+2xy+x^2=0.$

26. Obtain the equation of a straight line in terms of the intercepts on the axes. *(P. U. 1936)*

27. Plotting the logarithm of the pressure of a gas against the logarithm of its volume, the graph is found to be a straight line intersecting the axes at points whose distances from the origin are both +3. What is the relation between the pressure and the volume of the gas ? Name the law. *(P. U. 1930)*

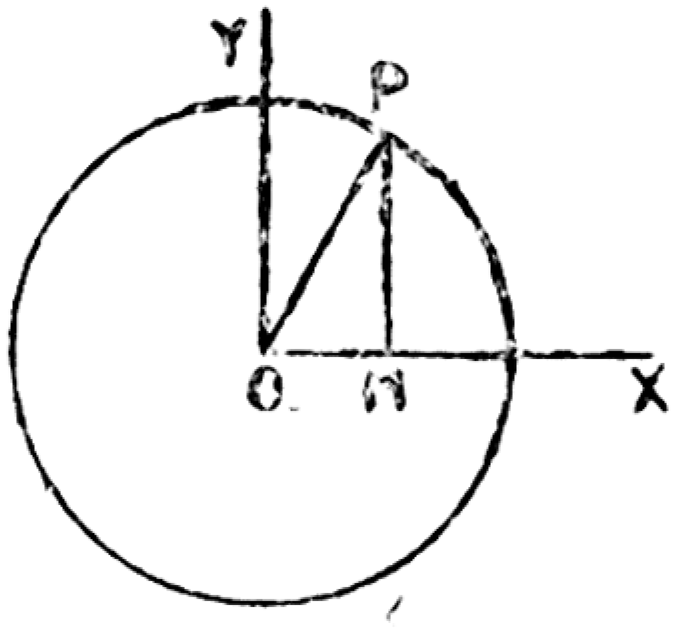
CHAPTER XXII

Equations to Circle, Parabola, Ellipse and Hyperbola

135. Def. A **circle** is the *locus of a point* which moves so that its *distance* from a *fixed point* is always constant.

The fixed point is called the **centre** of the circle.

136. *To find the equation to a circle having the origin as its centre.*



Draw a circle with O as centre and a as radius and take any point P on this circle.

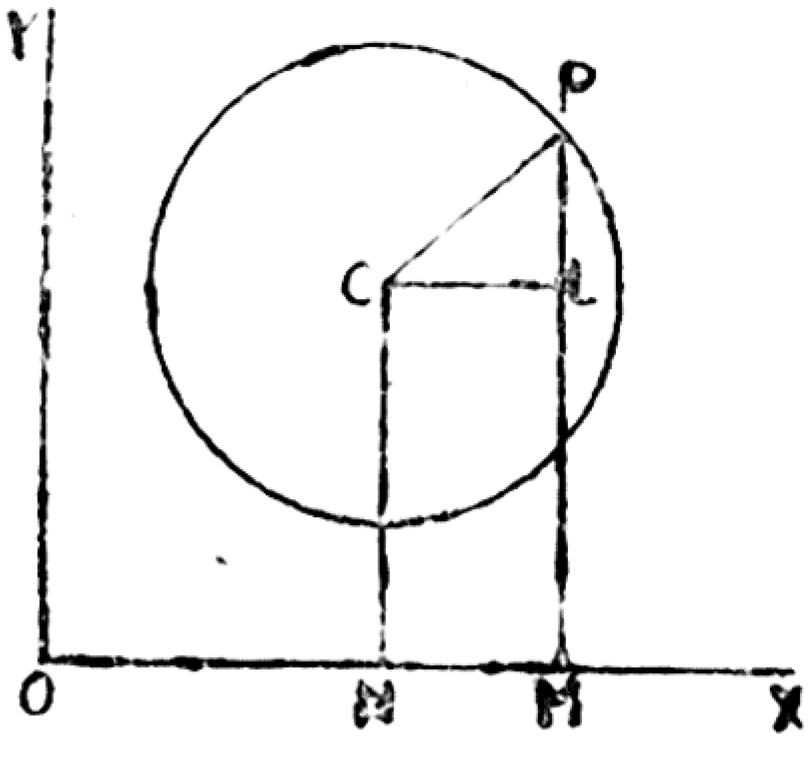
Join PO and draw PM the ordinate of P.

$$OM^2+PM^2=OP^2,$$

$$\therefore\quad x^2+y^2=a^2.$$

This is the required equation.

137. *To find the general equation to a circle.*



Let C (h, k) be the centre of a circle and a the radius.

Take any point P on the circle and draw PM and CN the ordinates of P and C. Draw CL parallel to OX to meet PM in L.

$$CL=NM=OM-ON=x-h,$$

$$PL=PM-LM=PM-CN=y-k$$

But $CL^2+PL^2=CP^2$.

$\therefore\quad (x-h)^2+(y-k)^2=a^2$ is the required equation.

138. Any equation which can be written in the form $(x-h)^2+(y-k)^2=a^2$ represents a circle with point (h, k) as centre and a as radius.

Consider the equation

$$x^2+y^2+2gx+2fy+c=0.$$

Add g^2+f^2 to both the sides and transpose c to the right.

$$x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c$$

i.e.,
$$(x+g)^2+(y+f)^2=\{\sqrt{g^2+f^2-c}\}^2$$

∴ the given equation represents a circle with $(-g, -f)$ as centre and $\sqrt{g^2+f^2-c}$ as radius.

139. Note the following points in the equation of the last article.

(*i*) **The equation is of the second degree.**

(*ii*) **Co-efficients of x^2 and y^2 are equal.**

(*iii*) **There is no term containing the product xy.**

If these conditions are satisfied we can always express a given equation in the form $(x-h)^2+(y-k)^2=a^2$ and therefore the equation represents a circle.

Example 1. Find the equation to the circle having its centre at (2, 3) and radius equal to 5.

$$h=2,\ k=3, \text{ and } a=5.$$

∴ the equation is $(x-2)^2+(y-3)^2=5^2$.

i.e., $$x^2+y^2-4x-6y-12=0.$$

Example 2. Find the centre and the radius of the circle represented by $2x^2+2y^2-2x-2y-1=0$.

Dividing all the terms by 2 and transposing the independent term to the right, we get

$$x^2+y^2-x-y=\tfrac{1}{2}.$$

Add $(\frac{1}{2})^2+(\frac{1}{2})^2$ to both the sides,

$$x^2-x+\tfrac{1}{4}+y^2-y+\tfrac{1}{4}=\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4}.$$

$$(x-\tfrac{1}{2})^2+(y-\tfrac{1}{2})^2=1^2,$$

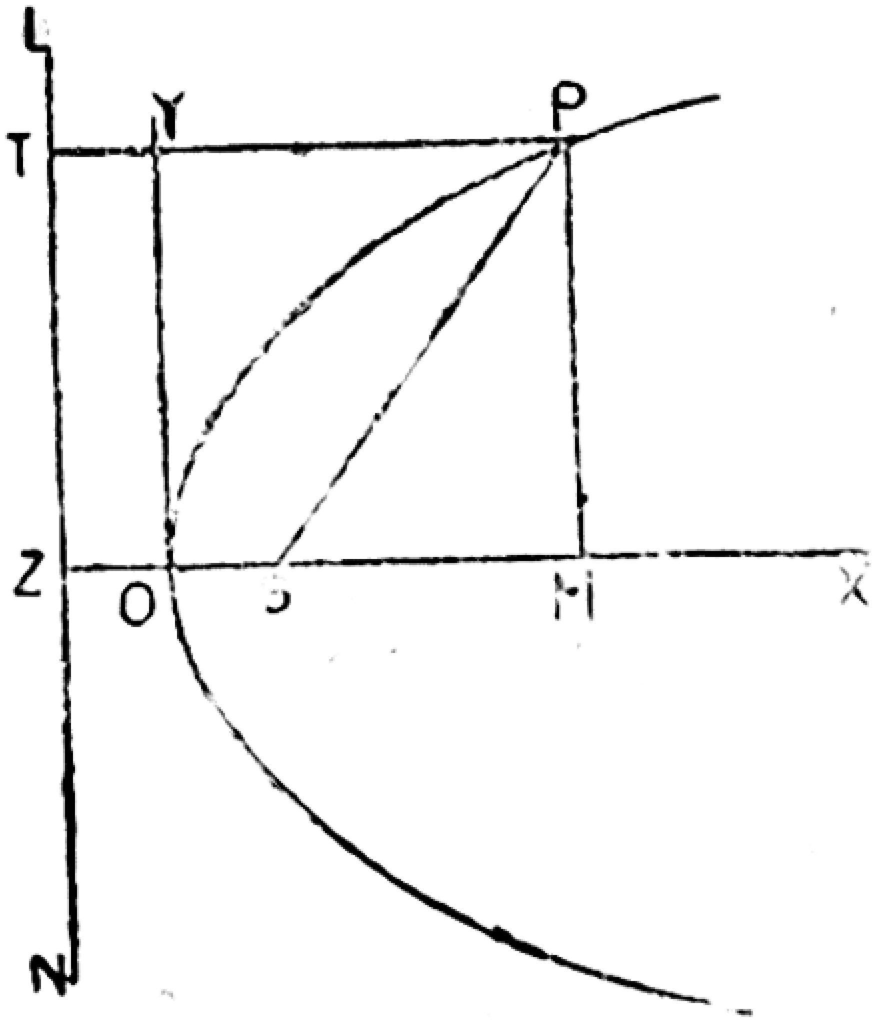
∴ the point $(\frac{1}{2}, \frac{1}{2})$ is the centre and 1 is the radius.

[NOTE. The quantity added to both the sides of the equation is the one necessary to complete the squares with x and y, and is therefore equal to the sum of the squares of the semi-co-efficients of x and y.]

THE PARABOLA

140. Def. A parabola is the *locus of a point* which moves so that its distance from *a fixed point* is always *equal* to its distance from a *fixed straight line*.

The fixed point (S) is called the **focus** and the fixed straight line LN is called the **directrix**.



The straight line XSZ perpendicular to the directrix and passing through the focus is called the **axis**.

The point O where the axis meets the parabola is called the **vertex**.

Imp

141. *To find the equation to a parabola referred to the vertex as origin and the axis as axis of x.*

Take any point P on the parabola. Join PS and draw PM and PT perpendiculars to the axis (OX) and the directix (LN) respectively.

From the vertex O draw OY as perpendicular to OX to represent the axis of Y.

Let $OS = a$

Point O lies on the parabola, $\therefore$ OS=OZ [Def.

Point P " " " " $\therefore$ PS=PT [Def.

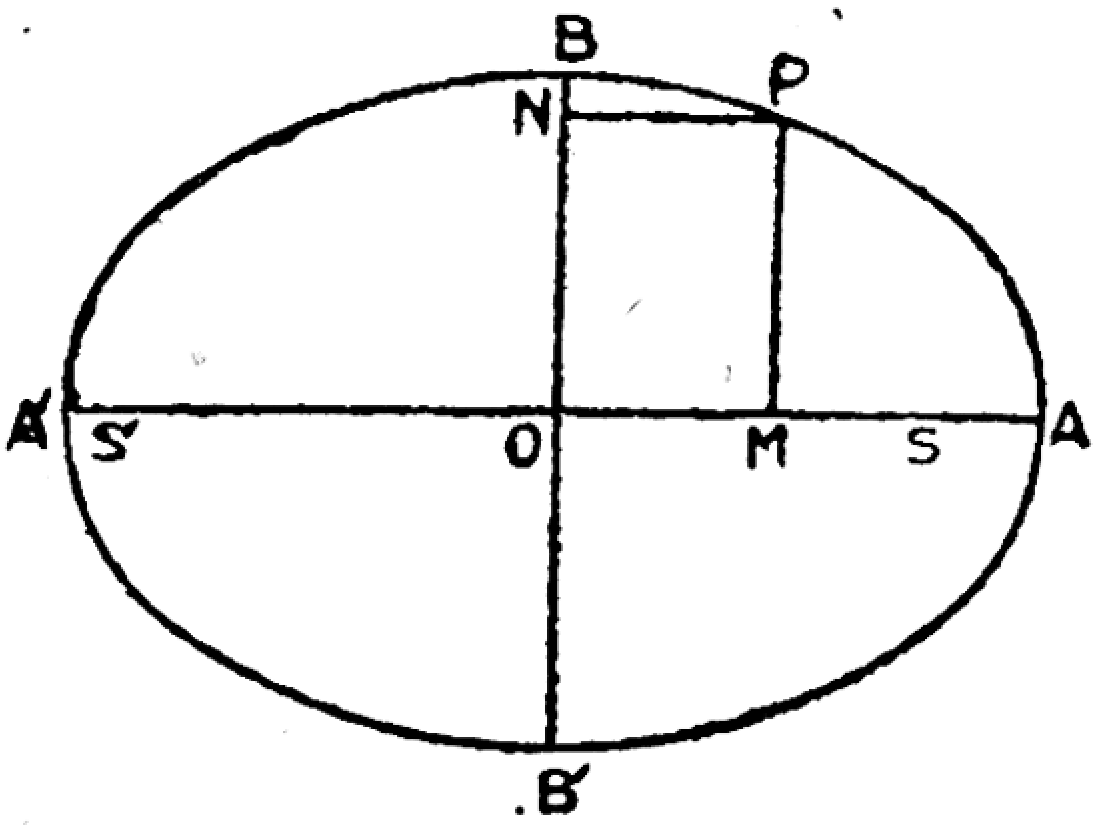
$$y^2 = PM^2 = PS^2 - SM^2 = PT^2 - SM^2$$
$$= ZM^2 - SM^2 = (OM + ZO)^2 - (OM - OS)^2$$
$$= (x+a)^2 - (x-a)^2 = 4ax.$$

$\therefore$ $y^2 = 4ax$ is the required equation. It may be written as $y^2 = mx$ where m is a constant (equal to $4a$).

x and y are always interchangeable in all standard equations and the figure rotates through a right angle if they are interchanged Thus $x^2=4ay$ will also represent a parabola whose axis lies on the axis of y.

THE ELLIPSE

142. Ellipse is the *locus of a point* which moves so that its *distance* from a *fixed point* bears a *constant ratio less than unity*, to its *distance* from a *fixed straight line*. As in case of **parabola** the point and the line are called the **focus** and the **directrix** respectively.



Ellipse, as proved in books on Geometry, is a closed curve having also a second focus S′. The line joining the foci is called the axis. There are two vertices A and A′.

Bisect AA′ at O and draw BOB′ perpendicular to AA′.

The curve is symmetrical about AA′ and BB′.

AA′ is called the major axis and BB′ the minor axis. The point O is called the centre.

143. *Equation to an ellipse referred to the centre as origin and the axes as axes of co-ordinates.*

If OA$=a$ and OB$=b$, the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ represents the relation between the co-ordinates of all the points on the ellipse referred to OA as axis of x and OB as axis of y.

[The proof directly follows from the relation $\frac{PN^2}{OA^2}+\frac{PM^2}{OB^2}=1$ as proved in books on geometry.]

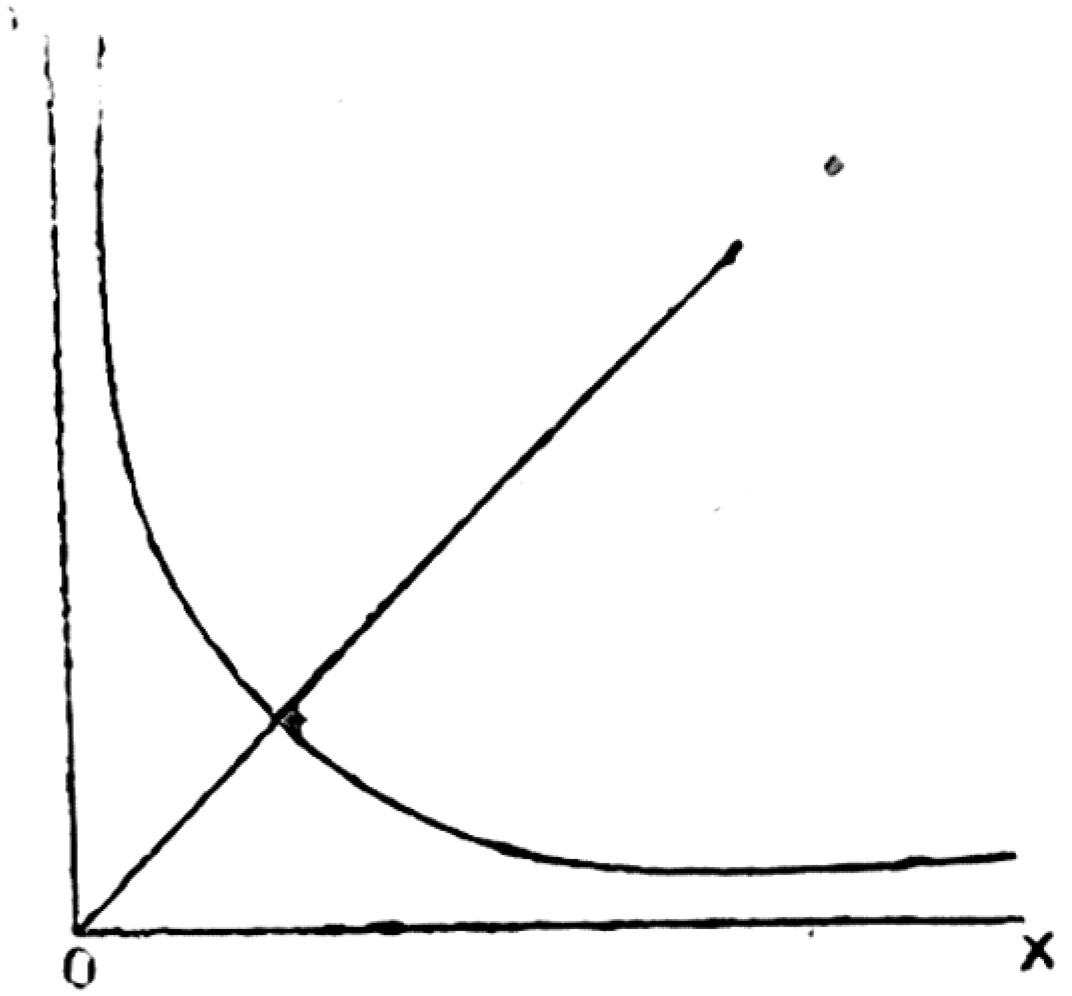
THE HYPERBOLA

144. Def.—Hyperbola is the *locus of a point* which moves so that its *distance* from a *fixed point* bears a *constant ratio, greater than unity,* to its distance from a *fixed straight line.*

As before, the point and the line are called the **focus** and the **directrix** respectively. The general equation to a hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

145. The equation to a particular kind of hyperbola known as **rectangular hyperbola referred to axes inclined at 45°** to its axis is $xy=c$ where c is some constant.



The equation $xy=c$ is of frequent occurrence in Physics and therefore the shape of its curve (rectangular parabola) is worth remembering (see figure).

The graph of P and V, where P denotes the pressure and V the volume of a given mass of a gas at a constant temperature, is a rectangular hyperbola because

P × V is constant [Boyle's law.]

Note. The parabola, the ellipse and the hyperbola are called **conic sections** or simply **conics.**

TRACING OF CURVES

146. If the equation of a curve can be expressed in one of the standard forms of equations to the straight line or the circle, data just necessary for drawing these can be obtained and the curve can be easily drawn on a graph paper.

(*a*) If the equation is of the first degree, it represents a straight line and if two points are found, the line can be drawn.

Example 3. Draw the line $4x+5y+7=0$.

Give any two convenient values to x and find the corresponding values of y.

$$\text{If } x=2,\ y=-3.$$

$$\text{If } x=-3,\ y=1.$$

Plot the points $(2, -3)$ and $(-3, 1)$ on a graph paper, join them by a straight line and produe it both ways. This will be the required line.

Sometimes it is more convenient to find the intercepts on the axes.

Example 4. Trace the curve of $\dfrac{y-33}{9}=\dfrac{x}{3}$.

$$\text{When } x=0,\ y=33.$$

$$\text{When } y=0,\ x=-11.$$

Take a point 33, along the axis of y, and a point -11 along the axis of x and join them.

(*b*) If the equation is quadratic, either it represents a pair of straight lines or a circle or some conic.

(*i*) If the equation is of the form $x^2+y^2+2gx+2fy+c=0$, it represents a circle with $(-g, -f)$ as centre and $\sqrt{g^2+f^2-c}$ as radius, and therefore the circle can be drawn.

(*ii*) If the equation is not of the form of the circle, try to split it up into two factors of the first degree and if successful draw the two straight lines separately.

(*iii*) If the equation does not come under (*i*) and (*ii*), give a series of values to x or y and find the corresponding values of y or x. Tabulate them, plot the points thus obtained on a graph paper and join them by a smooth curve. Parabola, ellipse etc. are also better drawn in this way.

(*c*) Curve of a degree higher than second should be treated as (*iii*) above.

Example 5. Trace the curve of $x^2+y^2+2x-8=0$.

This represents a circle (the co-efficients of $xy=0$).

The equation can be written as $(x+1)^2+(y-0)^2=9$.

$\therefore$ $(-1, 0)$ is the centre and 3 the radius.

Example 6. Trace the curve of $x^2+3xy+2y^2=0$.

This equation contains xy and the co-efficients of x^2 and y^2 are different, therefore it cannot represent a circle. It can however be factorised as is apparent, or by the formula of the quadratic equation

$$x=\frac{-3y+\sqrt{9y^2-8y^2}}{2}=\frac{-3y\pm y}{2}$$

$$\therefore\ x=-y \text{ or } -2y.$$

These are the equations of the two straight lines represented by the given quadratic equation.

Example 7. Trace the curve of $\frac{1}{y}+\frac{1}{x}=\frac{1}{8}$ for values of $x>8$.

When simplified the equation becomes $8x+8y=xy$.

The right hand side is the product of x and y, therefore it is a term of the second degree and hence the equation is quadratic. It can neither be split up into linear factors nor does it represent a circle.

Let us give a series of values greater than 8 to x and find the corresponding values of y.

$$\frac{1}{y}=\frac{1}{8}-\frac{1}{x}=\frac{x-8x}{8x}.$$

$$\therefore\quad y=\frac{8x}{x-8}.$$

x	10	12	14	16	18·7	24	40
y	40	24	18·7	16	14	12	10

In the given equation x and y are interchangeable, therefore after obtaining $x=y=16$, the values of x and y have been interchanged.

Plot these points on a graph paper and join them by a smooth curve.

Note. This is the curve for the distance of an object and the distance of the real image in a concave mirror or a convex lens $cf.\ \frac{1}{v}+\frac{1}{u}=\frac{1}{f}$.

Example 8. What will be the shape of the graph between T and l where $T=2\pi\sqrt{\frac{l}{g}}$ (simple pendulum) ?

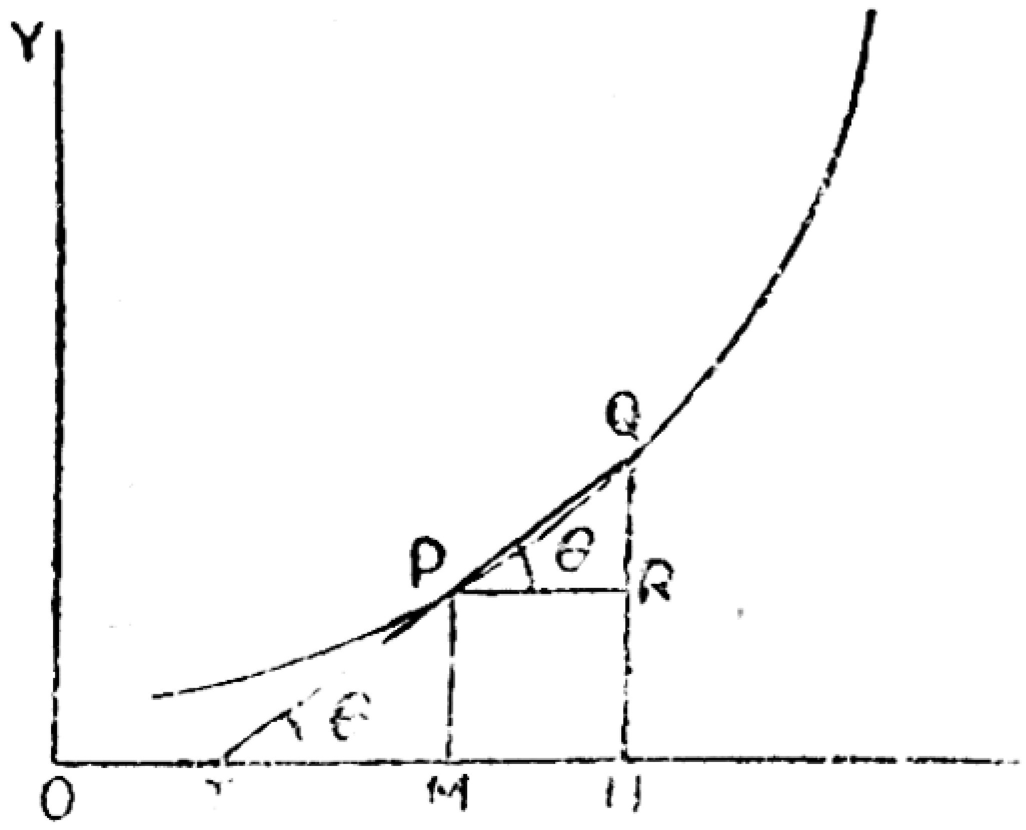
If T is represented along the axis of y and l along the axis of x, the formula in terms of the co-ordinates of a point on the curve becomes

$$y=2\pi\sqrt{\frac{x}{g}},\quad i.\,e.,\ y^2=\frac{4\pi^2}{g}x$$

4π, and g are constant, therefore $\frac{4\pi^2}{g}$ is also a constant.

Thus the equation is of the form $y^2=4ax$ which represents a parabola.

RATE OF CHANGE



147. Let above figure represent a graph between any two quantities. Consider two points P and Q close together on this curve. Draw PM and QN the ordinates of P and Q and draw PR parallel to the axis of x.

Let θ be the angle made by the straight line PQ with the axis of x.

While the values of the abscissa changes from OM to ON the ordinate changes from PM to QN, *viz.*, corresponding to a change MN in the abscissa, there is a change equal to QR in the ordinate.

Rate of change of one quantity with respect to another is defined as the ratio of the change in the first quantity to the corresponding change in the second.

Thus rate of change of ordinate with respect to abscissa is equal to $\frac{QR}{PR}=\tan\theta$.

When Q approaches P and ultimately coincides with it, the line PQ becomes the tangent at P and QR and PR become infinitely small.

Thus, the rate of change of ordinate with respect to abscissa at a given point on the curve is equal to the tangent of the angle made with the *axis* of x by the tangent to the curve at the point.

In the language of the differential calculus an infinitely small change in the ordinate is denoted by dy and the corresponding small change in the abscissa is denoted by dx, therefore $\frac{dy}{dx}=\tan\theta$ for any point on a graph.

Illustrations :—

(1) If the curve is a space time diagram (see some book on mechanics) *viz.*, graph between time as abscissa and distance travelled by a moving body as ordinate, the rate of change of ordinate with respect to abscissa means the velocity of the body. Thus, the velocity at a point is found by drawing a tangent to the curve at the given point and finding the tangent of the angle made by it with the axis of x.

(2) If the graph between time and velocity the rate of change means the acceleration and is found as in (1).

If the graph in (1) or (2) is a straight line, tangents to all the points on the curve make the same angle with the axis of x, therefore the velocity or acceleration is uniform.

EXAMPLES XXII

Find the equation to the circle—

1. Whose radius is 5 and centre is at $(-2, -1)$.

2. ,, ,, 2 ,, ,, ,, is at $(0, 2)$.

3. ,, ,, $\frac{7}{2}$,, ,, ,, is at $\left(-3, \frac{3}{2}\right)$.

4. ,, ,, 3 ,, ,, ,, is at $(3, 3)$,

Find the centres and radii of the following circles :—

5. $x^2+y^2+2x-4y=4$. **6.** $x^2+y^2+10x+12y=39$.

7. $x^2+y^2+x=0$. **8.** $x^2+y^2+2(3x+4y)=0$.

9. (*i*) Obtain the general equation of a circle.

(*ii*) Represent by a diagram the following circle in relation to its axes and determine the radius :—

$$x^2+y^2-a(2x+2y-a)=0. \qquad (P.\ U.\ 1930)$$

10. (*i*) Obtain the equation of a parabola, referred to vertex as the origin and the axis as axis of x.

(*P. U. 1929, 1931, 1938*)

(*ii*) An aeroplane moving horizontally at 60 ft. a second drops a bomb. If after t seconds the bomb has moved horizontally $60t$ ft. and fallen vertically $16t^2$ ft. show that the path described is a parabola, and determine the latus rectum. (latus rectum$=2a$). (*P. U. 1929*)

What will be the shape of the graph between—

11. v and t ; given $v=u+at$ (velocity and time in *uniformly accelerated motion*) ?

12. S and t in $S=\frac{1}{2}gt^2$ (*bodies falling freely* under the action of gravity) ?

13. F and a in $F=ma$ (*Newton's 2nd law*) ?

14. T^2 and l in $T=2\pi\sqrt{\frac{l}{g}}$ (*Simple pendulum*)?

15. P and h in $\frac{P}{W}=\frac{h}{l}$ (*Inclined* plane) ?

16. P and V in $PV=k$ (*Boyle's law*) ?

17. log P and log V in $PV=k$ (*Boyle's law*) ?

18 V_t and t in $V_t=V_0\,(1+\gamma t)$ (*Charles's law*) ?

19. $u+v$ and uv in $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$ (*lenses and mirrors*) ?

20. d^3 and $\tan\theta$ in $\frac{2M}{d^3}=H\tan\theta$ (*Deflection magnetometer*)?

21. E and C in $\frac{E}{R}=C$ (*Ohm's law*) ?

22. R and $\tan\theta$ in $\frac{E}{R}=k\tan\theta$ (*Tangent galvanometer*) ?

23. R and $\frac{1}{\tan\theta}$ in $\frac{E}{R}=k\tan\theta$ (*Tangent galvanometer*) ?

24. Transform the relation between the length and time period of a simple pendulum, so as to show that it is of the form of the equation of a parabola. *(P. U. 1927)*

[**Hint** : See Example 8 on page 151]

25. What is the shape of the curve relating the area of a circle and its radius ? *(P. U. 1935)*

26. A stone is thrown in a horizontal direction from the top of a tower. Show that its path is a parabola. Find the equation and the latus rectum of the parabola if the stone is thrown with a velocity of 350 cm. per second at a place where g is equal to 980 cm. per second per second. *(P. U. 1933)*

27. Determine the form of the curve showing the relation between L, the length of a simple pendulum, and T, its period of oscillation. Also prove that the graph showing the relation between L and T^2 is a straight line. *(P. U. 1934)*

4014 1206
9 22
4023 1228

APPENDIX I

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

·8897
2

5839

APPENDIX II

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

1807
2
1805

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6540	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

98,63.
9
987,2

ANSWERS

Examples I. (Page 5).

5. 99·4 cm. **6.** 79 therms and 1.

Examples II. (Page 11).

1. ·0006% approx. **2.** ·0012% approx.
5. ·0004486%. **6.** ·6714%.

Examples III. (Page 24).

1. 3, 1, 4, 5, 4. **2.** ·5785, 2·5785, $\bar{2}$·5785, $\bar{1}$·5785, $\bar{5}$·5785.
3. 2984, 29840, 298·4, ·2984. **4.** 1·5050. **5.** 1·5474.
6. $\bar{1}$·4419. **7.** 1·3380. **8.** ·6990.
9. ·7573. **10.** 13·60. **11.** 15210. **12.** 2·429.
13. ·03868. **14.** 4·354. **15.** ·9811. **16.** ·2510
17. ·3219. **18.** ·08085. **19.** ·9145. **20.** ·06974.
21. $C=k \tan\theta$ (tangent galvanometer) **22.** $L=KP^2$.
23. ·605. **24.** square root. **25.** 1·794.
26. 99·40. **27.** 161·4.

Examples IV. (Page 28).

1. 4, 1. **2.** 1, -2. **3.** 3, -2. **4.** -10, -7.
5. 8, -42. **6.** $\frac{3}{2}$, $\frac{2}{3}$. **7.** $\frac{4}{3}$, $-\frac{3}{4}$. **8.** 5, $-\frac{32}{7}$.
9. $\frac{1}{13}$, $\frac{1}{60}$. **10.** $\frac{3}{2}$, $-\frac{1}{2}$. **11.** 4, $\frac{7}{5}$. **12.** 5, $-\frac{5}{2}$.
13. 8, $\frac{5}{2}$. **14.** $-\frac{1}{2}$, $\frac{-73}{50}$. **15.** $\pm\sqrt{6}$. **16.** 16, 2.
17. 2, $\frac{1}{3}$. **18.** 24, $\frac{42}{5}$. **19.** 3, $\frac{-4}{3}$. **20.** 1, $\frac{13}{8}$.
21. 0, $\frac{4}{3}$. **22.** 1, $\frac{-3}{7}$. **23.** 6, $\frac{40}{13}$.

2.99114

24. $\left(\frac{-11+\sqrt{13}}{6}\right)a.$ **25.** $a+b+c.$ **26.** $1, \frac{2b}{a-b}$

27. $\frac{1}{3}\{a+b+c\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}\}.$

28. $\frac{a^2+b^2\pm\sqrt{(a^2-b^2)^2+4abc^2}}{2ab}.$

Examples V. (Page 33).

1. $2^{2n}, -\frac{1}{2^{2n}}.$ **2.** $\frac{1}{9}, \frac{25}{4}.$ **3.** 2, 3. **4.** $\pm 1, \pm\sqrt{3}.$

5. 4, -1. **6.** $\frac{125}{64}$, 8. **7.** 15, 5. **8.** 1, $\frac{1}{2}$.

9. 79, 6. **10.** 21, 6 ; $-6, -21$. **11.** 14, 11 ; $-14, -11$.

12. 120 yds. and 121 yds. **13.** 3. **14.** 16 ; £5.

Examples VI. (Page 37).

1. Real and unequal. **2.** Real and equal.

3. Real and unequal. **4.** Imaginary and unequal.

5. Real and unequal. **6.** $3x^2-13x+3=0.$

7. $25x^2-25x+6=0.$ **8.** $x^2-4x+1=0.$

9. $x^2-2x-1=0.$ **10.** $x^2+x-6=0.$

11. $x^2-7x+12=0.$ **12.** $x^2-15x+36=0.$

13. $x^2-25x-100=0.$ **14.** 20. **15.** 18.

16. 5. **17.** 2. **18.** 8. **19.** 4.

20. 4. **21.** $\frac{31}{8}$. **23.** 2·2 sec. **24.** 6 sec.

25. (*a*) ·4706 sec., 2·654 sec. (*b*) 4·511 sec. **27.** 5 sec.

28. $\frac{5}{4}, \frac{5}{2}$ sec. **29.** 2, 5 sec.

Examples VII. (Page 43)

1. 40. **2.** -11. **3.** 4. **4.** $7+2r$.

5. 800. **6.** $3\frac{3}{4}$. **7.** $25\frac{1}{3}$. **8.** -18. **9.** 425.

10. $\frac{n}{2(1-m)}\left\{2+(n-3)\sqrt{m}\right\}.$ **11.** 94, $860\sqrt{3}$.

13. $\frac{n}{2}\left\{x(n+1)--2y(n+2)\right\}.$ **15.** $\frac{-1}{4}, \frac{-3}{4}, \frac{-5}{4}$ etc.

14. 3 and 5. **16.** $\frac{13}{3}, \frac{11}{3}, \frac{9}{3}, \frac{7}{3}, \frac{5}{3}, \frac{3}{3}$.

17. 1640. **18.** $-28{\cdot}5, -12{\cdot}45, 3{\cdot}6$…… **19.** 10.

20. 8. **21.** 16, 15. **22.** 25. **23.** 17.

24. 6, 10, 14. **25.** n^2. **26.** $1, -2, -5$……

27. 2, 4, 6, 8. **28.** 5, 10, 15. **29.** $6p-1$.

30. 2. **33.** 9 ; 16. **34.** $m+n-r$. **35.** 30,000.

36. $\frac{n(n+1)}{2}$. **37.** 236, 240, 244……472.

Examples VIII. (Page 50).

1. 243. **2.** $\frac{1}{64}$. **3.** 243.

4. $(3+2\sqrt{2})\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^{n-1}$ **5.** -682. **6.** $\frac{3367}{1536}$.

7. 23. **8.** $\frac{1}{9}\left(10-\frac{1}{10^9}\right)$ **9.** $\frac{16}{3}$. **10.** $\frac{3(3+\sqrt{3})}{2}$.

11. $\frac{1}{3}$. **12.** $\frac{27}{58}$. **13.** $\frac{16}{3}$, 8, 12, 18, 27. **14.** $\frac{3}{2}$, 1, $\frac{2}{3}$.

15. Any G. P., with $r=2$. **17.** 16, 24, 36………

18. 3, 6, 12. **19.** 2. **22.** $6, -3, \frac{3}{2}$,………

24. 4, 16. **52,** $\frac{a(1-r^n)}{1-r}$. **26.** 4·76 lb. per sq. inch.

Examples IX. (Page 56)

1. (*i*) n^2 (*ii*) $n(n+1)$. **2.** $\frac{x(x^{\overset{n}{2}}-1)}{(x-1)}+\frac{n(n+2)}{8}a$.

3. $\frac{a^n(a^{2n}-1)}{a^2-1}+\frac{ab(a^nb^n-1)}{ab-1}$. **4.** $\frac{31}{80}$. **5.** 1220.

6. $n(a+\overline{n-1}\;d)$; $\frac{n(n+1)(3a+\overline{n-1}\times 2d)}{6}$.

7. $\frac{1}{2}(2^{n+1}+1)-2^{n+1}$. **8.** $\frac{4}{9}\{\frac{10}{9}(10^n-1)-n\}$. **9.** 60.

10, $b+2pc-c$ **11.** 256. **12.** $\frac{1}{2}$. **14.** 71, 74, 228. **15.** 1,220.

Examples X. (Page 63)

1. $1-18a^2+135a^4-540a^6+1215a^8-1458a^{10}+729a^{12}$.

2. $32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$.

3. $625-2000x+2400x^2-1280x^3+256x^4$.

4. $1-\frac{10}{a}+\frac{45}{a^2}-\frac{120}{a^3}+\frac{210}{a^4}-\frac{252}{a^5}+\frac{210}{a^6}-\frac{120}{a^7}+\frac{45}{a^8}-\frac{10}{a^9}+\frac{1}{a^{10}}$.

5. $y^5+5y^3+10y+\frac{10}{y}+\frac{5}{y^3}+\frac{1}{y^5}$.

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6. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$; $1+\frac{1}{2}x$.

7. $1+x+\frac{x^2}{6}-\frac{x^3}{54}$; $1+x$.

8. $1-x+\frac{2}{3}x^2-\frac{10}{27}x^3$; $1-x$. **9.** $1+x+2x^2+\frac{14}{3}x^3$, $1+x$.

10. $1+3x+\frac{15}{2}x^2+\frac{35}{2}x^3$; $1+3x$. **11.** $1+2x+3x^2+4x^3$; $1+2x$.

12. $-35750x^{10}$. **13.** $\frac{1120}{81}a^4c^4$. **14.** $\frac{77}{256}x^{10}$.

15. $\frac{16b^4}{243a^5}$. **16.** $\frac{-429}{16}a^{14}$. **17.** ·20123. **18.** 999·99.

19. 1·0100. **20.** 1,30,321.

Examples XI. (Page 69)

1. ·6501. **2.** 1·967. **3.** ·4508. **4.** ·2618.

5. 38°12′. **6.** 40° 7′. **7.** 86°26′. **8.** 67°30′

9. 105°. **10.** ·0044 inch. **11.** 112. **12.** $\frac{\pi}{12}$, $\frac{\pi}{3}$, $\frac{7\pi}{12}$.

13. 2 miles 1600 yds. **14.** 31·416 miles per hour. **15.** 25°37.

Examples XII (Page 77).

1. (*i*) cos (*ii*) tan (*iii*) sin (*iv*) tan
(*v*) all (*vi*) tan (*vii*) all (*viii*) cos

19. $\pm\frac{7}{24}$, $\pm\frac{25}{24}$. **20.** $\pm\frac{\sqrt{5}}{2}$, $\pm\sqrt{5}$.

21. $\pm\frac{\sqrt{48}}{7}$, $\pm\frac{1}{\sqrt{48}}$. **22.** $\pm\frac{m^2-n^2}{m^2+n^2}$, $\pm\frac{m^2+n^2}{2mn}$.

23. 3. **24.** 48°. **25.** 2·7. **26.** 26°.

Examples XIII (Page 82).

1. $\frac{1}{4}$. **2.** 0. **3.** $-\frac{1}{4}$. **4.** 2.

5. 1. **6.** 9. **7.** $\frac{3}{2}$. **8.** 30°.
9. 45°. **10.** 30° ; 45°. **11.** 30°. **12.** 75°.
13. 0°. **14.** 135°.

Examples XIV (Page 87)

1. $\frac{-\sqrt{3}}{2}$. **2.** −1. **3.** 1. **4.** $\frac{-2}{\sqrt{3}}$. **5.** $\frac{1}{\sqrt{2}}$. **6.** −cos A.
7. −cos A. **8.** tan A. **13.** $\cot^2$ A. **14.** 3. **15.** −1.

Examples XV (Page 92)

2. ·7711. **3.** ·9625. **4.** ·9435. **5.** ·21927. **6.** ·0910.
7. −·6858. **8.** −·9178. **9.** 13°34′, 166°26′. **10.** 43°20′.
11. 72°43′. **12.** 82°57′. **13.** 108°45′. **14.** 126°41′.

Examples XVI (Page 100).

3. $\frac{33}{65}$. **4.** $\frac{980}{2501}, \frac{2499}{2501}$. **5.** $\frac{-85}{36}$.
6. $\frac{1}{3}$. **7.** $\frac{5}{13}, \frac{12}{13}$. **8.** $\frac{1}{7}$.

9. $\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\tan 75° = 2+\sqrt{3}$.

Examples XVII (Page 109).

1. a=2, A=30°, B=60°.
2. c=12·65, A=18°26′, B=71°34′.
3. a=20, A=B=45°. **4.** b=4, $a=4\sqrt{3}$, B=30°.
5. a=4, $c=4\sqrt{2}$, B=45°.
6. a=6 472, b=4·702, B=36°.
7. c=39·39, a=30·43, A=50°6.
8. c=353, A=39°36′, B=50°24′.
9. a=1817, c=5254, B=69°46′.
10. b=·0477, c=·0622, B=50°.
11. a=654·3, A=64°13, B=25°47′.
12. a=6·273, A=64°27′, B=25°33′.
13. a=8390, c=8420, A=85°·27.
14. 14·14, 20. **15.** 42·79 ft. **16.** 70°34′.
17. 164·1 ft. **18.** 524·0 ft. **19.** Both 70·99 ft.
20. 48·7 ft. **21.** 300 ft. **22.** 106 miles per hour.

Examples XVIII (Page 129)

1. 60°. 2. $a=15$, B$=27°48'$, A$=90°12'$. **3.** 6.
4. 8·697, 16°42′. **5.** 18·25, 9°28′. **6.** 13·90, 34°7′.
7. 12·76, 71°41′. **8.** $\cos\theta$ comes to be 2 or 3 which is impossible, $\therefore$ no value of θ can satisfy this relation.
9. 2·155 metres. **10.** 95·5 yds. **11.** ·4925. **12.** 234·4 sec.
14. $\sqrt{10}$. 18. ·366 cm. **19.** 13·89 poundals ; 36°55′.
20. 3 lb. 21. $\sqrt{2}$.

Examples XX (Page 136)

1. 4·47. 2. 8·24. **3.** 5·66. **4.** 5·00.
5. 5·39. **6.** 14·87. **7.** $3\pm2\sqrt{15}$.

Examples XXI (Page 142)

1. $x-y+1=0$. **2.** $x-y\sqrt{3}-2\sqrt{3}=0$. **3.** 135°.
4 120°58′. **5.** $x+y=0$. **6.** $7y+10x=11$.
7. $5x+3y+15=0$. 8. $3x+2y-6=0$.
9. $2x+3y-2=0$. **10.** $x+y+4=0$.
11. $2x-3y=4$, $y-3x=1$, $x+2y=2$.
14. $\dfrac{x}{15/4}+\dfrac{y}{5}=1$. **15.** $\dfrac{x}{-2}+\dfrac{y}{-5}=1$. **16.** $\dfrac{x}{\frac{1}{2}}+\dfrac{y}{-\frac{2}{3}}=1$.
17. $(-3, -5)$. **18.** $(1, -2)$. **19.** $y=x+1$.
20. $2y+3x=0$. **24.** $x=2y$ and $x=-2y$.

25. Coincident lines $x+y=0$. $\dfrac{x}{a}+\dfrac{y}{b}=1$. **27.** PV$=1{,}000$.

Examples XXII (Page 153)

1. $x^2+y^2+4x+2y-20=0$. **2.** $x^2+y^2-4y=0$.
3. $x^2+y^2+6x-3y-1=0$. **4.** $x^2+y^2-6x-6y+9=0$.
5. $(-1, 2)$; 3. **6.** $(-5, -6)$; 10.
7. $(-\frac{1}{2}, 0)$; $\frac{1}{2}$. **8.** $(-3, -4)$; 5.
9. Draw a circle with centre at (a, a) and radius a.
10. (*i*) Art. 141 (*ii*) 225 ft. **11.** Straight line.
12. One branch of parabola. **13.** Straight line.
14. Straight line. **15.** Straight line.
16. Rectangular hyperbola. **17.** Straight line.
18. Straight line. **19** Straight line.
20. Rectangular hyperbola. **21.** Straight line.
22. Rectangular hyperbola. **23.** Straight line. **25.** Parabola.
26. $x^2=250\,y$, latus rectum$=250$ cm.

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